is flowing directly south. It is observed that the true direction of the boat is directly east.

(a) Express the velocity of the boat relative to the water as a vector in component form.

(b) Find the speed of the water and the true speed of the boat.

66. Velocity A woman walks due west on the deck of an ocean liner at 2 mi/h. The ocean liner is moving due north at a speed of 25 mi/h. Find the speed and direction of the woman relative to the surface of the water.

67–72 ■ Equilibrium of Forces The forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at the same point P are said to be in equilibrium if the resultant force is zero, that is, if $\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = 0$. Find **(a)** the resultant forces acting at P, and **(b)** the additional force required (if any) for the forces to be in equilibrium.

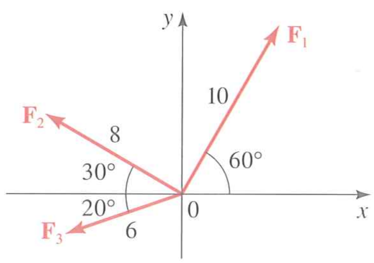
67. $\mathbf{F}_1 = \langle 2, 5 \rangle$, $\mathbf{F}_2 = \langle 3, -8 \rangle$

68. $\mathbf{F}_1 = \langle 3, -7 \rangle$, $\mathbf{F}_2 = \langle 4, -2 \rangle$, $\mathbf{F}_3 = \langle -7, 9 \rangle$

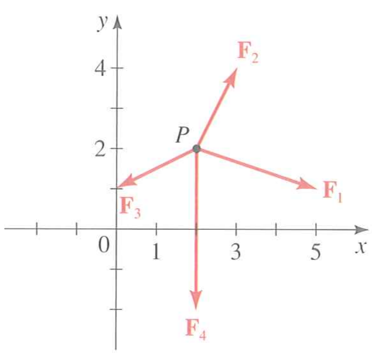
69. $\mathbf{F}_1 = 4\mathbf{i} - \mathbf{j}$, $\mathbf{F}_2 = 3\mathbf{i} - 7\mathbf{j}$, $\mathbf{F}_3 = -8\mathbf{i} + 3\mathbf{j}$, $\mathbf{F}_4 = \mathbf{i} + \mathbf{j}$

70. $\mathbf{F}_1 = \mathbf{i} - \mathbf{j}$, $\mathbf{F}_2 = \mathbf{i} + \mathbf{j}$, $\mathbf{F}_3 = -2\mathbf{i} + \mathbf{j}$

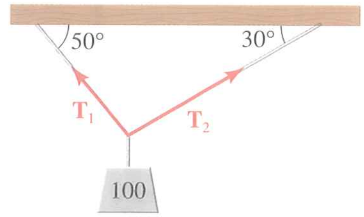
71.



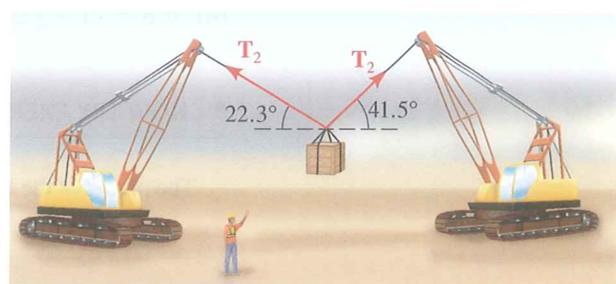
72.



73. Equilibrium of Tensions A 100-lb weight hangs from a string as shown in the figure. Find the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ in the string.

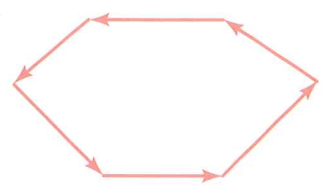


74. Equilibrium of Tensions The cranes in the figure are lifting an object that weighs 18,278 lb. Find the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$.



DISCOVERY ■ DISCUSSION ■ WRITING

75. Vectors That Form a Polygon Suppose that n vectors can be placed head to tail in the plane so that they form a polygon. (The figure shows the case of a hexagon.) Explain why the sum of these vectors is $\mathbf{0}$.



9.2 The Dot Product

The Dot Product of Vectors ▶ The Component of **u** Along **v** ▶ The Projection of **u** Onto **v** ▶ Work

In this section we define an operation on vectors called the dot product. This concept is especially useful in calculus and in applications of vectors to physics and engineering.

▼ The Dot Product of Vectors

We begin by defining the dot product of two vectors.

DEFINITION OF THE DOT PRODUCT

If $\mathbf{u} = \langle a_1, b_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2 \rangle$ are vectors, then their **dot product**, denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined by

$$\mathbf{u} \cdot \mathbf{v} = a_1 a_2 + b_1 b_2$$

Thus to find the dot product of **u** and **v**, we multiply corresponding components and add. The dot product is *not* a vector; it is a real number, or scalar.

EXAMPLE 1 | Calculating Dot Products

(a) If $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 5 \rangle$ then

$$\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(5) = 2$$

(b) If $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 6\mathbf{j}$, then

$$\mathbf{u} \cdot \mathbf{v} = (2)(5) + (1)(-6) = 4$$

NOW TRY EXERCISES **5(a)** AND **11(a)**

The proofs of the following properties of the dot product follow easily from the definition.

PROPERTIES OF THE DOT PRODUCT

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (a\mathbf{v})$
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
4. $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$

PROOF We prove only the last property. The proofs of the others are left as exercises. Let $\mathbf{u} = \langle a, b \rangle$. Then

$$\mathbf{u} \cdot \mathbf{u} = \langle a, b \rangle \cdot \langle a, b \rangle = a^2 + b^2 = |\mathbf{u}|^2$$

Let **u** and **v** be vectors, and sketch them with initial points at the origin. We define the **angle θ between u and v** to be the smaller of the angles formed by these representations of **u** and **v** (see Figure 1). Thus $0 \le \theta \le \pi$. The next theorem relates the angle between two vectors to their dot product.

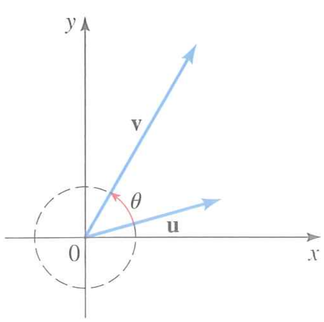


FIGURE 1

THE DOT PRODUCT THEOREM

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$$

PROOF Applying the Law of Cosines to triangle AOB in Figure 2 gives

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos \theta$$

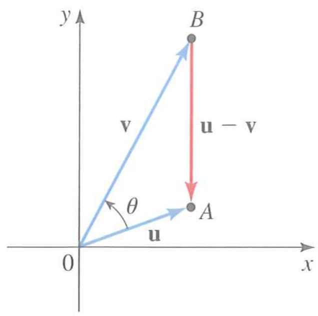


FIGURE 2

Using the properties of the dot product, we write the left-hand side as follows:

$$\begin{aligned} |\mathbf{u} - \mathbf{v}|^2 &= (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 \end{aligned}$$

Equating the right-hand sides of the displayed equations, we get

$$\begin{aligned} |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta \\ -2(\mathbf{u} \cdot \mathbf{v}) &= -2|\mathbf{u}||\mathbf{v}|\cos\theta \\ \mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}||\mathbf{v}|\cos\theta \end{aligned}$$

This proves the theorem. ■

The Dot Product Theorem is useful because it allows us to find the angle between two vectors if we know the components of the vectors. The angle is obtained simply by solving the equation in the Dot Product Theorem for $\cos\theta$. We state this important result explicitly.

ANGLE BETWEEN TWO VECTORS

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

EXAMPLE 2 | Finding the Angle Between Two Vectors

Find the angle between the vectors $\mathbf{u} = \langle 2, 5 \rangle$ and $\mathbf{v} = \langle 4, -3 \rangle$.

SOLUTION By the formula for the angle between two vectors we have

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{(2)(4) + (5)(-3)}{\sqrt{4 + 25}\sqrt{16 + 9}} = \frac{-7}{5\sqrt{29}}$$

Thus the angle between **u** and **v** is

$$\theta = \cos^{-1}\left(\frac{-7}{5\sqrt{29}}\right) \approx 105.1°$$

NOW TRY EXERCISES **5(b)** AND **11(b)** ■

Two nonzero vectors **u** and **v** are called **perpendicular**, or **orthogonal**, if the angle between them is $\pi/2$. The following theorem shows that we can determine whether two vectors are perpendicular by finding their dot product.

ORTHOGONAL VECTORS

Two nonzero vectors **u** and **v** are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

PROOF If **u** and **v** are perpendicular, then the angle between them is $\pi/2$, so

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\frac{\pi}{2} = 0$$

Conversely, if $\mathbf{u} \cdot \mathbf{v} = 0$, then

$$|\mathbf{u}||\mathbf{v}| \cos\theta = 0$$

Since **u** and **v** are nonzero vectors, we conclude that $\cos\theta = 0$, so $\theta = \pi/2$. Thus **u** and **v** are orthogonal. ■

EXAMPLE 3 | Checking Vectors for Perpendicularity

Determine whether the vectors in each pair are perpendicular.

(a) $\mathbf{u} = \langle 3, 5 \rangle$ and $\mathbf{v} = \langle 2, -8 \rangle$ **(b)** $\mathbf{u} = \langle 2, 1 \rangle$ and $\mathbf{v} = \langle -1, 2 \rangle$

SOLUTION

(a) $\mathbf{u} \cdot \mathbf{v} = (3)(2) + (5)(-8) = -34 \neq 0$, so **u** and **v** are not perpendicular.

(b) $\mathbf{u} \cdot \mathbf{v} = (2)(-1) + (1)(2) = 0$, so **u** and **v** are perpendicular.

NOW TRY EXERCISES **15** AND **17**

The Component of u Along v

The **component of u along v** (or the **component of u in the direction of v**) is defined to be

$$|\mathbf{u}| \cos \theta$$

Note that the component of **u** along **v** is a scalar, not a vector.

where θ is the angle between **u** and **v**. Figure 3 gives a geometric interpretation of this concept. Intuitively, the component of **u** along **v** is the magnitude of the portion of **u** that points in the direction of **v**. Notice that the component of **u** along **v** is negative if $\pi/2 < \theta \leq \pi$.

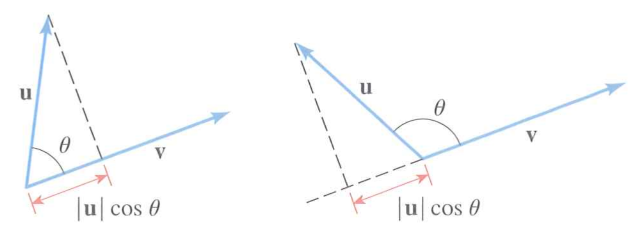


FIGURE 3

In analyzing forces in physics and engineering, it's often helpful to express a vector as a sum of two vectors lying in perpendicular directions. For example, suppose a car is parked on an inclined driveway as in Figure 4. The weight of the car is a vector **w** that points directly downward. We can write

$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$

where **u** is parallel to the driveway and **v** is perpendicular to the driveway. The vector **u** is the force that tends to roll the car down the driveway, and **v** is the force experienced by the surface of the driveway. The magnitudes of these forces are the components of **w** along **u** and **v**, respectively.

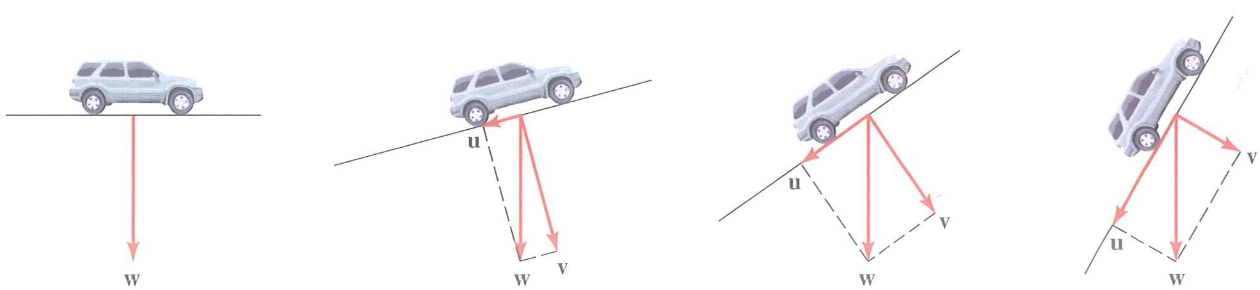


FIGURE 4

EXAMPLE 4 | Resolving a Force into Components

A car weighing 3000 lb is parked on a driveway that is inclined 15° to the horizontal, as shown in Figure 5.

(a) Find the magnitude of the force required to prevent the car from rolling down the driveway.

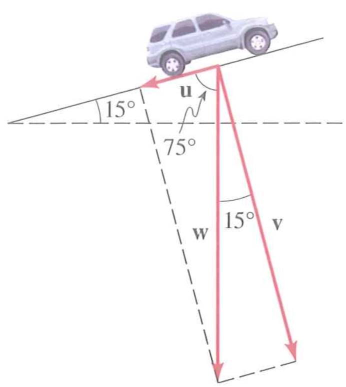


FIGURE 5

(b) Find the magnitude of the force experienced by the driveway due to the weight of the car.

SOLUTION The car exerts a force **w** of 3000 lb directly downward. We resolve **w** into the sum of two vectors **u** and **v**, one parallel to the surface of the driveway and the other perpendicular to it, as shown in Figure 5.

(a) The magnitude of the part of the force **w** that causes the car to roll down the driveway is

$$|\mathbf{u}| = \text{component of } \mathbf{w} \text{ along } \mathbf{u} = 3000 \cos 75° \approx 776$$

Thus the force needed to prevent the car from rolling down the driveway is about 776 lb.

(b) The magnitude of the force exerted by the car on the driveway is

$$|\mathbf{v}| = \text{component of } \mathbf{w} \text{ along } \mathbf{v} = 3000 \cos 15° \approx 2898$$

The force experienced by the driveway is about 2898 lb.

NOW TRY EXERCISE 49

The component of **u** along **v** can be computed by using dot products:

$$|\mathbf{u}| \cos \theta = \frac{|\mathbf{v}||\mathbf{u}| \cos \theta}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$$

We have shown the following.

CALCULATING COMPONENTS

The component of **u** along **v** is $\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$.

EXAMPLE 5 | Finding Components

Let $\mathbf{u} = \langle 1, 4 \rangle$ and $\mathbf{v} = \langle -2, 1 \rangle$. Find the component of **u** along **v**.

SOLUTION We have

$$\text{component of } \mathbf{u} \text{ along } \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{(1)(-2) + (4)(1)}{\sqrt{4 + 1}} = \frac{2}{\sqrt{5}}$$

NOW TRY EXERCISE 25

▼ The Projection of u Onto v

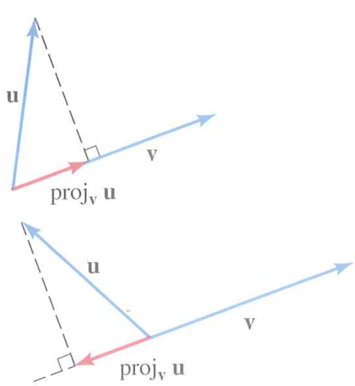


FIGURE 6

Figure 6 shows representations of the vectors **u** and **v**. The projection of **u** onto **v**, denoted by $\text{proj}_\mathbf{v} \mathbf{u}$, is the vector *parallel* to **v** and whose *length* is the component of **u** along **v** as shown in Figure 6. To find an expression for $\text{proj}_\mathbf{v} \mathbf{u}$, we first find a unit vector in the direction of **v** and then multiply it by the component of **u** along **v**:

$$\text{proj}_\mathbf{v} \mathbf{u} = (\text{component of } \mathbf{u} \text{ along } \mathbf{v})(\text{unit vector in direction of } \mathbf{v})$$

$$= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}\right) \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right) \mathbf{v}$$

We often need to **resolve** a vector **u** into the sum of two vectors, one parallel to **v** and one orthogonal to **v**. That is, we want to write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to **v** and $\mathbf{u}_2$ is orthogonal to **v**. In this case, $\mathbf{u}_1 = \text{proj}_\mathbf{v} \mathbf{u}$ and $\mathbf{u}_2 = \mathbf{u} - \text{proj}_\mathbf{v} \mathbf{u}$ (see Exercise 43).

CALCULATING PROJECTIONS

The **projection of u onto v** is the vector $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ given by

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

If the vector **u** is **resolved** into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to **v** and $\mathbf{u}_2$ is orthogonal to **v**, then

$$\mathbf{u}_1 = \text{proj}_{\mathbf{v}}\,\mathbf{u} \qquad \text{and} \qquad \mathbf{u}_2 = \mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$$

EXAMPLE 6 | Resolving a Vector into Orthogonal Vectors

Let $\mathbf{u} = \langle -2, 9\rangle$ and $\mathbf{v} = \langle -1, 2\rangle$.

(a) Find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

(b) Resolve **u** into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to **v** and $\mathbf{u}_2$ is orthogonal to **v**.

SOLUTION

(a) By the formula for the projection of one vector onto another we have

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \qquad \text{Formula for projection}$$

$$= \left(\frac{\langle -2, 9\rangle\cdot\langle -1, 2\rangle}{(-1)^2 + 2^2}\right)\langle -1, 2\rangle \qquad \text{Definition of } \mathbf{u} \text{ and } \mathbf{v}$$

$$= 4\langle -1, 2\rangle = \langle -4, 8\rangle$$

(b) By the formula in the preceding box we have $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where

$$\mathbf{u}_1 = \text{proj}_{\mathbf{v}}\,\mathbf{u} = \langle -4, 8\rangle \qquad \text{From part (a)}$$

$$\mathbf{u}_2 = \mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u} = \langle -2, 9\rangle - \langle -4, 8\rangle = \langle 2, 1\rangle$$

NOW TRY EXERCISE 29

Work

One use of the dot product occurs in calculating work. In everyday use, the term *work* means the total amount of effort required to perform a task. In physics, *work* has a technical meaning that conforms to this intuitive meaning. If a constant force of magnitude F moves an object through a distance d along a straight line, then the **work** done is

$$W = Fd \qquad \text{or} \qquad \text{work} = \text{force} \times \text{distance}$$

If F is measured in pounds and d in feet, then the unit of work is a foot-pound (ft-lb). For example, how much work is done in lifting a 20-lb weight 6 ft off the ground? Since a force of 20 lb is required to lift this weight and since the weight moves through a distance of 6 ft, the amount of work done is

$$W = Fd = (20)(6) = 120 \text{ ft-lb}$$

This formula applies only when the force is directed along the direction of motion. In the general case, if the force **F** moves an object from P to Q, as in Figure 7, then only the component of the force in the direction of $\mathbf{D} = \overrightarrow{PQ}$ affects the object. Thus the effective magnitude of the force on the object is

$$\text{component of } \mathbf{F} \text{ along } \mathbf{D} = |\mathbf{F}|\cos\theta$$

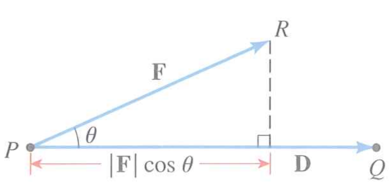


FIGURE 7

So the work done is

$$W = \text{force} \times \text{distance} = (|\mathbf{F}|\cos\theta)|\mathbf{D}| = |\mathbf{F}||\mathbf{D}|\cos\theta = \mathbf{F}\cdot\mathbf{D}$$

We have derived the following simple formula for calculating work.

> **WORK**
>
> The **work** W done by a force $\mathbf{F}$ in moving along a vector $\mathbf{D}$ is
>
> $$W = \mathbf{F} \cdot \mathbf{D}$$

EXAMPLE 7 | Calculating Work

A force is given by the vector $\mathbf{F} = \langle 2, 3 \rangle$ and moves an object from the point $(1, 3)$ to the point $(5, 9)$. Find the work done.

SOLUTION The displacement vector is

$$\mathbf{D} = \langle 5 - 1, 9 - 3 \rangle = \langle 4, 6 \rangle$$

So the work done is

$$W = \mathbf{F} \cdot \mathbf{D} = \langle 2, 3 \rangle \cdot \langle 4, 6 \rangle = 26$$

If the unit of force is pounds and the distance is measured in feet, then the work done is 26 ft-lb.

NOW TRY EXERCISE **35**

EXAMPLE 8 | Calculating Work

A man pulls a wagon horizontally by exerting a force of 20 lb on the handle. If the handle makes an angle of 60° with the horizontal, find the work done in moving the wagon 100 ft.

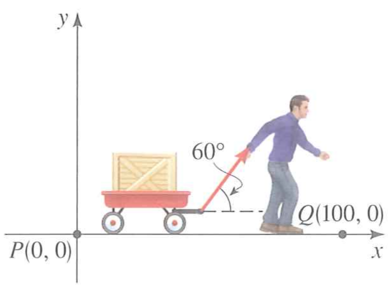


FIGURE 8

SOLUTION We choose a coordinate system with the origin at the initial position of the wagon (see Figure 8). That is, the wagon moves from the point $P(0, 0)$ to the point $Q(100, 0)$. The vector that represents this displacement is

$$\mathbf{D} = 100\,\mathbf{i}$$

The force on the handle can be written in terms of components (see Section 9.1) as

$$\mathbf{F} = (20 \cos 60°)\mathbf{i} + (20 \sin 60°)\mathbf{j} = 10\,\mathbf{i} + 10\sqrt{3}\,\mathbf{j}$$

Thus the work done is

$$W = \mathbf{F} \cdot \mathbf{D} = (10\,\mathbf{i} + 10\sqrt{3}\,\mathbf{j}) \cdot (100\,\mathbf{i}) = 1000 \text{ ft-lb}$$

NOW TRY EXERCISE **47**

9.2 EXERCISES

CONCEPTS

1–2 ■ Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ be nonzero vectors in the plane, and let θ be the angle between them.

1. The dot product of $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a} \cdot \mathbf{b} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

The dot product of two vectors is a ___________, not a vector.

2. The angle θ satisfies

$$\cos \theta = \frac{\_\_\_\_\_}{\_\_\_\_\_}$$

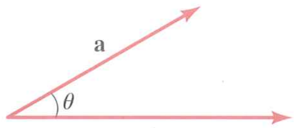
So if $\mathbf{a} \cdot \mathbf{b} = 0$, the vectors are ______________.

3. **(a)** The component of **a** along **b** is the scalar $|a|\cos\theta$ and can be expressed in terms of the dot product as

_____________. Sketch this component in the figure below.

(b) The projection of **a** onto **b** is the vector

$\text{proj}_{\mathbf{b}}\, \mathbf{a} =$ _____________. Sketch this projection in the figure below.



4. The work done by a force **F** in moving an object along a vector **D** is $W =$ _____________.

SKILLS

5–14 ■ Find **(a)** $\mathbf{u}\cdot\mathbf{v}$ and **(b)** the angle between **u** and **v** to the nearest degree.

5. $\mathbf{u} = \langle 2, 0\rangle,\quad \mathbf{v} = \langle 1, 1\rangle$

6. $\mathbf{u} = \mathbf{i} + \sqrt{3}\mathbf{j},\quad \mathbf{v} = -\sqrt{3}\mathbf{i} + \mathbf{j}$

7. $\mathbf{u} = \langle 2, 7\rangle,\quad \mathbf{v} = \langle 3, 1\rangle$

8. $\mathbf{u} = \langle -6, 6\rangle,\quad \mathbf{v} = \langle 1, -1\rangle$

9. $\mathbf{u} = \langle 3, -2\rangle,\quad \mathbf{v} = \langle 1, 2\rangle$

10. $\mathbf{u} = 2\mathbf{i} + \mathbf{j},\quad \mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

11. $\mathbf{u} = -5\mathbf{j},\quad \mathbf{v} = -\mathbf{i} - \sqrt{3}\mathbf{j}$

12. $\mathbf{u} = \mathbf{i} + \mathbf{j},\quad \mathbf{v} = \mathbf{i} - \mathbf{j}$

13. $\mathbf{u} = \mathbf{i} + 3\mathbf{j},\quad \mathbf{v} = 4\mathbf{i} - \mathbf{j}$

14. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j},\quad \mathbf{v} = -2\mathbf{i} - \mathbf{j}$

15–20 ■ Determine whether the given vectors are perpendicular.

15. $\mathbf{u} = \langle 6, 4\rangle,\quad \mathbf{v} = \langle -2, 3\rangle$

16. $\mathbf{u} = \langle 0, -5\rangle,\quad \mathbf{v} = \langle 4, 0\rangle$

17. $\mathbf{u} = \langle -2, 6\rangle,\quad \mathbf{v} = \langle 4, 2\rangle$

18. $\mathbf{u} = 2\mathbf{i},\quad \mathbf{v} = -7\mathbf{j}$

19. $\mathbf{u} = 2\mathbf{i} - 8\mathbf{j},\quad \mathbf{v} = -12\mathbf{i} - 3\mathbf{j}$

20. $\mathbf{u} = 4\mathbf{i},\quad \mathbf{v} = -\mathbf{i} + 3\mathbf{j}$

21–24 ■ Find the indicated quantity, assuming $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$, and $\mathbf{w} = 3\mathbf{i} + 4\mathbf{j}$.

21. $\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

22. $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w})$

23. $(\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} - \mathbf{v})$

24. $(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})$

25–28 ■ Find the component of **u** along **v**.

25. $\mathbf{u} = \langle 4, 6\rangle,\quad \mathbf{v} = \langle 3, -4\rangle$

26. $\mathbf{u} = \langle -3, 5\rangle,\quad \mathbf{v} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$

27. $\mathbf{u} = 7\mathbf{i} - 24\mathbf{j},\quad \mathbf{v} = \mathbf{j}$

28. $\mathbf{u} = 7\mathbf{i},\quad \mathbf{v} = 8\mathbf{i} + 6\mathbf{j}$

29–34 ■ **(a)** Calculate $\text{proj}_{\mathbf{v}}\,\mathbf{u}$. **(b)** Resolve **u** into $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to **v** and $\mathbf{u}_2$ is orthogonal to **v**.

29. $\mathbf{u} = \langle -2, 4\rangle,\quad \mathbf{v} = \langle 1, 1\rangle$

30. $\mathbf{u} = \langle 7, -4\rangle,\quad \mathbf{v} = \langle 2, 1\rangle$

31. $\mathbf{u} = \langle 1, 2\rangle,\quad \mathbf{v} = \langle 1, -3\rangle$

32. $\mathbf{u} = \langle 11, 3\rangle,\quad \mathbf{v} = \langle -3, -2\rangle$

33. $\mathbf{u} = \langle 2, 9\rangle,\quad \mathbf{v} = \langle -3, 4\rangle$

34. $\mathbf{u} = \langle 1, 1\rangle,\quad \mathbf{v} = \langle 2, -1\rangle$

35–38 ■ Find the work done by the force **F** in moving an object from P to Q.

35. $\mathbf{F} = 4\mathbf{i} - 5\mathbf{j};\quad P(0, 0), Q(3, 8)$

36. $\mathbf{F} = 400\mathbf{i} + 50\mathbf{j};\quad P(-1, 1), Q(200, 1)$

37. $\mathbf{F} = 10\mathbf{i} + 3\mathbf{j};\quad P(2, 3), Q(6, -2)$

38. $\mathbf{F} = -4\mathbf{i} + 20\mathbf{j};\quad P(0,10), Q(5,25)$

39–42 ■ Let **u**, **v**, and **w** be vectors, and let a be a scalar. Prove the given property.

39. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

40. $(a\mathbf{u})\cdot\mathbf{v} = a(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(a\mathbf{v})$

41. $(\mathbf{u} + \mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

42. $(\mathbf{u} - \mathbf{v})\cdot(\mathbf{u} + \mathbf{v}) = |\mathbf{u}|^2 - |\mathbf{v}|^2$

43. Show that the vectors $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ and $\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$ are orthogonal.

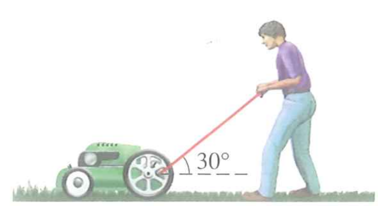
44. Evaluate $\mathbf{v}\cdot\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

APPLICATIONS

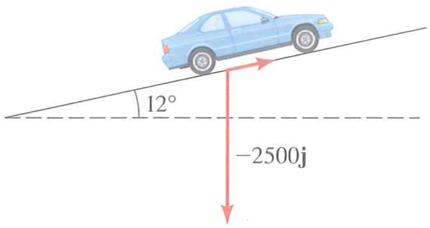
45. Work The force $\mathbf{F} = 4\mathbf{i} - 7\mathbf{j}$ moves an object 4 ft along the x-axis in the positive direction. Find the work done if the unit of force is the pound.

46. Work A constant force $\mathbf{F} = \langle 2, 8\rangle$ moves an object along a straight line from the point $(2, 5)$ to the point $(11, 13)$. Find the work done if the distance is measured in feet and the force is measured in pounds.

47. Work A lawn mower is pushed a distance of 200 ft along a horizontal path by a constant force of 50 lb. The handle of the lawn mower is held at an angle of 30° from the horizontal (see the figure). Find the work done.



48. Work A car drives 500 ft on a road that is inclined 12° to the horizontal, as shown in the following figure. The car weighs 2500 lb. Thus gravity acts straight down on the car with a constant force $\mathbf{F} = -2500\mathbf{j}$. Find the work done by the car in overcoming gravity.

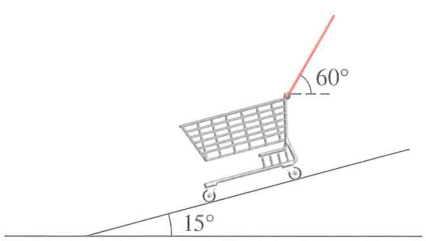


49. **Force** A car is on a driveway that is inclined 25° to the horizontal. If the car weighs 2755 lb, find the force required to keep it from rolling down the driveway.

50. **Force** A car is on a driveway that is inclined 10° to the horizontal. A force of 490 lb is required to keep the car from rolling down the driveway.
 (a) Find the weight of the car.
 (b) Find the force the car exerts against the driveway.

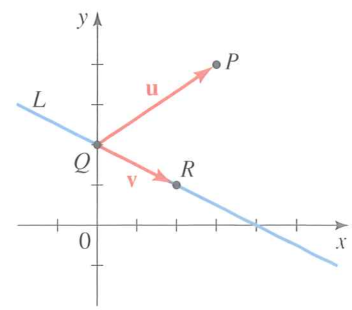
51. **Force** A package that weighs 200 lb is placed on an inclined plane. If a force of 80 lb is just sufficient to keep the package from sliding, find the angle of inclination of the plane. (Ignore the effects of friction.)

52. **Force** A cart weighing 40 lb is placed on a ramp inclined at 15° to the horizontal. The cart is held in place by a rope inclined at 60° to the horizontal, as shown in the figure. Find the force that the rope must exert on the cart to keep it from rolling down the ramp.



DISCOVERY ■ DISCUSSION ■ WRITING

53. **Distance from a Point to a Line** Let L be the line $2x + 4y = 8$ and let P be the point $(3, 4)$.
 (a) Show that the points $Q(0, 2)$ and $R(2, 1)$ lie on L.
 (b) Let $\mathbf{u} = \overrightarrow{QP}$ and $\mathbf{v} = \overrightarrow{QR}$, as shown in the figure. Find $\mathbf{w} = \text{proj}_{\mathbf{v}}\, \mathbf{u}$.
 (c) Sketch a graph that explains why $|\mathbf{u} - \mathbf{w}|$ is the distance from P to L. Find this distance.
 (d) Write a short paragraph describing the steps you would take to find the distance from a given point to a given line.



DISCOVERY PROJECT **Sailing Against the Wind**

In this project we study how sailors use the method of taking a zigzag path, or *tacking*, to sail against the wind. You can find the project at the book companion website: **www.stewartmath.com**

9.3 Three-Dimensional Coordinate Geometry

The Three-Dimensional Rectangular Coordinate System ▶ Distance Formula in Three Dimensions ▶ The Equation of a Sphere

To locate a point in a plane, two numbers are necessary. We know that any point in the Cartesian plane can be represented as an ordered pair (a, b) of real numbers, where a is the x-coordinate and b is the y-coordinate. In three-dimensional space, a third dimension is added, so any point in space is represented by an ordered triple (a, b, c) of real numbers.

The Three-Dimensional Rectangular Coordinate System

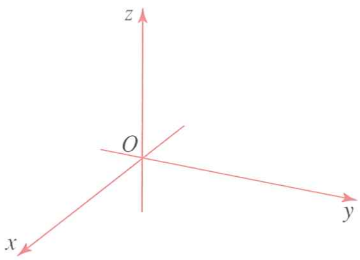


FIGURE 1 Coordinate axes

To represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical, and we draw the orientation of the axes as in Figure 1.

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 2(a). The xy-plane is the plane that contains the x- and y-axes; the yz-plane is the plane that contains the y- and z-axes; the xz-plane is the plane that contains the x- and z-axes. These three coordinate planes divide space into eight parts, called **octants**.

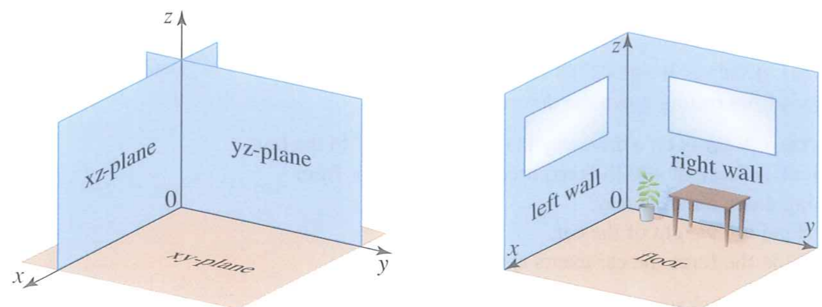


FIGURE 2 (a) Coordinate planes (b) Coordinate "walls"

Because people often have difficulty visualizing diagrams of three-dimensional figures, you may find it helpful to do the following (see Figure 2(b)). Look at any bottom corner of a room and call the corner the origin. The wall on your left is in the xz-plane, the wall on your right is in the yz-plane, and the floor is in the xy-plane. The x-axis runs along the intersection of the floor and the left wall; the y-axis runs along the intersection of the floor and the right wall. The z-axis runs up from the floor toward the ceiling along the intersection of the two walls.

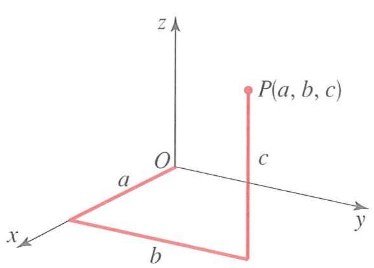


FIGURE 3 Point $P(a, b, c)$

Now any point P in space can be located by a unique **ordered triple** of real numbers (a, b, c), as shown in Figure 3. The first number a is the x-coordinate of P, the second number b is the y-coordinate of P, and the third number c is the z-coordinate of P. The set of all ordered triples $\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$ forms the **three-dimensional rectangular coordinate system**.

EXAMPLE 1 | Plotting Points in Three Dimensions

Plot the points $(2, 4, 7)$ and $(-4, 3, -5)$.

SOLUTION The points are plotted in Figure 4.

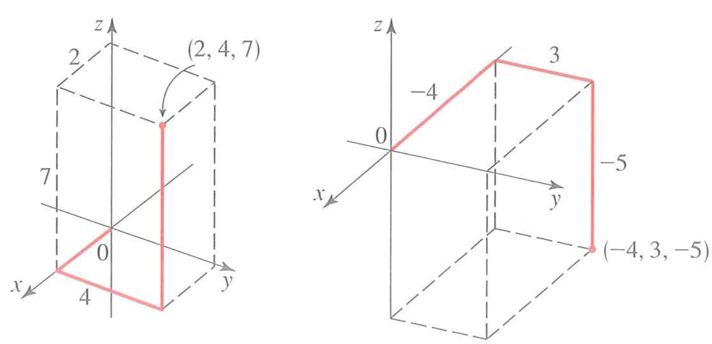


FIGURE 4

NOW TRY EXERCISE 3(a) ■

In two-dimensional geometry the graph of an equation involving x and y is a *curve* in the plane. In three-dimensional geometry an equation in x, y, and z represents a *surface* in space.