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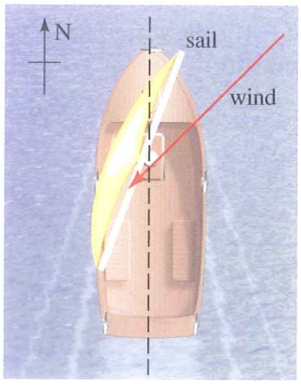
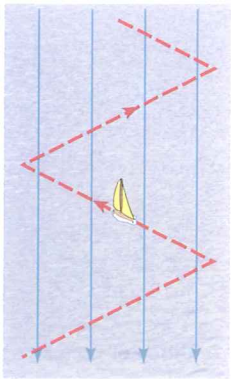
CHAPTER 9

VECTORS IN TWO AND THREE DIMENSIONS

Many real-world quantities are described mathematically by just one number: their "size" or magnitude. For example, quantities such as mass, volume, distance, and temperature are described by their magnitude. But many other real-world quantities involve both magnitude *and* direction. Such quantities are described mathematically by vectors. For example, if you push a car with a certain force, the direction in which you push on the car is important; you get different results if you push the car forward, backward, or perhaps sideways. So force is a vector. The result of several forces acting on an object can be evaluated by using vectors. For example, we'll see how we can combine the vector forces of wind and water on the sails and hull of a sailboat to find the direction in which the boat will sail. Analyzing these vector forces helps sailors to sail against the wind by tacking. (See the Discovery Project *Sailing Against the Wind* referenced on page 597.)



Vector forces



Tacking against the wind

9.1 Vectors in Two Dimensions

Geometric Description of Vectors ▶ Vectors in the Coordinate Plane ▶ Using Vectors to Model Velocity and Force

In applications of mathematics, certain quantities are determined completely by their magnitude—for example, length, mass, area, temperature, and energy. We speak of a length of 5 m or a mass of 3 kg; only one number is needed to describe each of these quantities. Such a quantity is called a **scalar**.

On the other hand, to describe the displacement of an object, two numbers are required: the *magnitude* and the *direction* of the displacement. To describe the velocity of a moving object, we must specify both the *speed* and the *direction* of travel. Quantities such as displacement, velocity, acceleration, and force that involve magnitude as well as direction are called *directed quantities*. One way to represent such quantities mathematically is through the use of *vectors*.

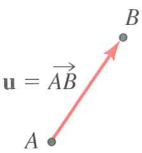


FIGURE 1

Geometric Description of Vectors

A **vector** in the plane is a line segment with an assigned direction. We sketch a vector as shown in Figure 1 with an arrow to specify the direction. We denote this vector by $\overrightarrow{AB}$. Point A is the **initial point**, and B is the **terminal point** of the vector $\overrightarrow{AB}$. The length of the line segment AB is called the **magnitude** or **length** of the vector and is denoted by $|\overrightarrow{AB}|$. We use boldface letters to denote vectors. Thus we write $\mathbf{u} = \overrightarrow{AB}$.

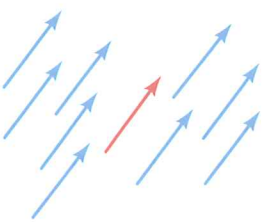


FIGURE 2

Two vectors are considered **equal** if they have equal magnitude and the same direction. Thus all the vectors in Figure 2 are equal. This definition of equality makes sense if we think of a vector as representing a displacement. Two such displacements are the same if they have equal magnitudes and the same direction. So the vectors in Figure 2 can be thought of as the *same* displacement applied to objects in different locations in the plane.

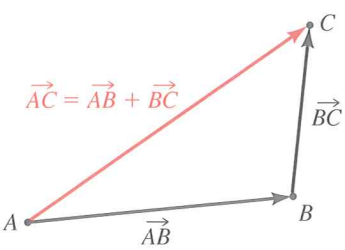


FIGURE 3

If the displacement $\mathbf{u} = \overrightarrow{AB}$ is followed by the displacement $\mathbf{v} = \overrightarrow{BC}$, then the resulting displacement is $\overrightarrow{AC}$ as shown in Figure 3. In other words, the single displacement represented by the vector $\overrightarrow{AC}$ has the same effect as the other two displacements together. We call the vector $\overrightarrow{AC}$ the **sum** of the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, and we write $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$. (The **zero vector**, denoted by $\mathbf{0}$, represents no displacement.) Thus to find the sum of any two vectors $\mathbf{u}$ and $\mathbf{v}$, we sketch vectors equal to $\mathbf{u}$ and $\mathbf{v}$ with the initial point of one at the terminal point of the other (see Figure 4(a)). If we draw $\mathbf{u}$ and $\mathbf{v}$ starting at the same point, then $\mathbf{u} + \mathbf{v}$ is the vector that is the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ shown in Figure 4(b).

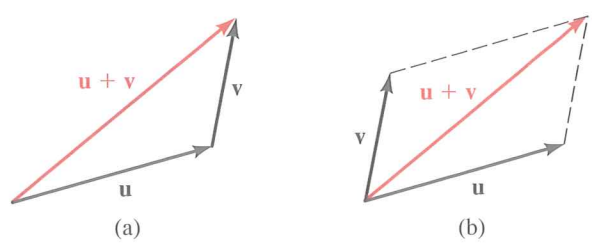


FIGURE 4 Addition of vectors

If a is a real number and $\mathbf{v}$ is a vector, we define a new vector $a\mathbf{v}$ as follows: The vector $a\mathbf{v}$ has magnitude $|a|\,|\mathbf{v}|$ and has the same direction as $\mathbf{v}$ if $a > 0$ and the opposite direction if $a < 0$. If $a = 0$, then $a\mathbf{v} = \mathbf{0}$, the zero vector. This process is called **multiplication of a vector by a scalar**. Multiplying a vector by a scalar has the effect of stretching or shrinking the vector. Figure 5 shows graphs of the vector $a\mathbf{v}$ for different values of a. We write the vector $(-1)\mathbf{v}$ as $-\mathbf{v}$. Thus $-\mathbf{v}$ is the vector with the same length as $\mathbf{v}$ but with the opposite direction.

The **difference** of two vectors **u** and **v** is defined by $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$. Figure 6 shows that the vector $\mathbf{u} - \mathbf{v}$ is the other diagonal of the parallelogram formed by **u** and **v**.

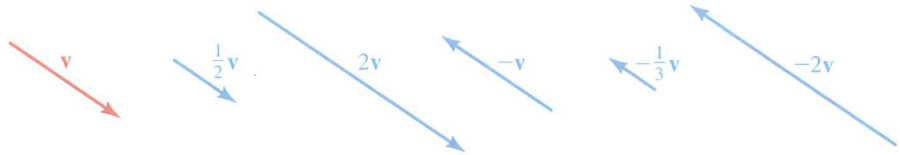


FIGURE 5 Multiplication of a vector by a scalar

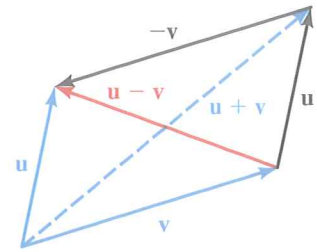


FIGURE 6 Subtraction of vectors

Vectors in the Coordinate Plane

So far, we've discussed vectors geometrically. By placing a vector in a coordinate plane, we can describe it analytically (that is, by using components). In Figure 7(a), to go from the initial point of the vector **v** to the terminal point, we move a units to the right and b units upward. We represent **v** as an ordered pair of real numbers.

Note the distinction between the *vector* $\langle a, b\rangle$ and the *point* (a, b).

$$\mathbf{v} = \langle a, b\rangle$$

where a is the **horizontal component** of **v** and b is the **vertical component** of **v**. Remember that a vector represents a magnitude and a direction, not a particular arrow in the plane. Thus the vector $\langle a, b\rangle$ has many different representations, depending on its initial point (see Figure 7(b)).

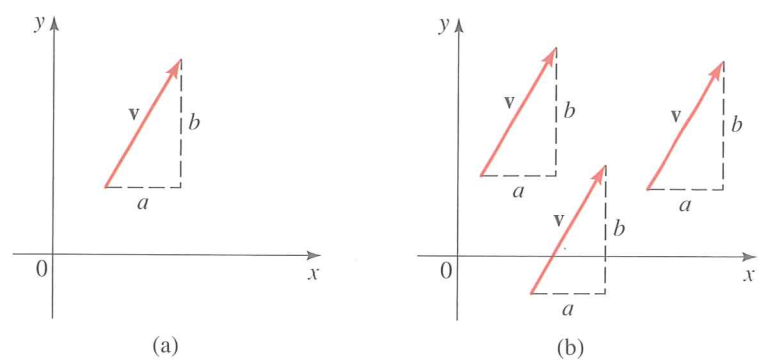


FIGURE 7

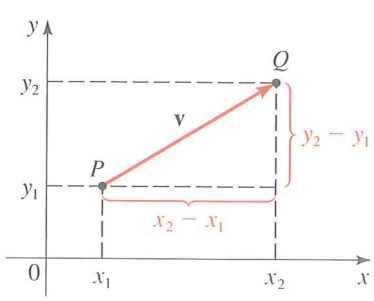


FIGURE 8

Using Figure 8, we can state the relationship between a geometric representation of a vector and the analytic one as follows.

COMPONENT FORM OF A VECTOR

If a vector **v** is represented in the plane with initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$, then

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1\rangle$$

EXAMPLE 1 | Describing Vectors in Component Form

(a) Find the component form of the vector **u** with initial point $(-2, 5)$ and terminal point $(3, 7)$.

(b) If the vector $\mathbf{v} = \langle 3, 7\rangle$ is sketched with initial point $(2, 4)$, what is its terminal point?

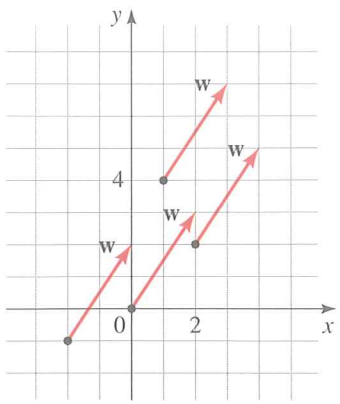


FIGURE 9

(c) Sketch representations of the vector $\mathbf{w} = \langle 2, 3\rangle$ with initial points at $(0, 0)$, $(2, 2)$, $(-2, -1)$, and $(1, 4)$.

SOLUTION

(a) The desired vector is

$$\mathbf{u} = \langle 3 - (-2), 7 - 5\rangle = \langle 5, 2\rangle$$

(b) Let the terminal point of $\mathbf{v}$ be (x, y). Then

$$\langle x - 2, y - 4\rangle = \langle 3, 7\rangle$$

So $x - 2 = 3$ and $y - 4 = 7$, or $x = 5$ and $y = 11$. The terminal point is $(5, 11)$.

(c) Representations of the vector $\mathbf{w}$ are sketched in Figure 9.

NOW TRY EXERCISES **11, 19,** AND **23**

We now give analytic definitions of the various operations on vectors that we have described geometrically. Let's start with equality of vectors. We've said that two vectors are equal if they have equal magnitude and the same direction. For the vectors $\mathbf{u} = \langle a_1, b_1\rangle$ and $\mathbf{v} = \langle a_2, b_2\rangle$, this means that $a_1 = a_2$ and $b_1 = b_2$. In other words, two vectors are **equal** if and only if their corresponding components are equal. Thus all the arrows in Figure 7(b) represent the same vector, as do all the arrows in Figure 9.

Applying the Pythagorean Theorem to the triangle in Figure 10, we obtain the following formula for the magnitude of a vector.

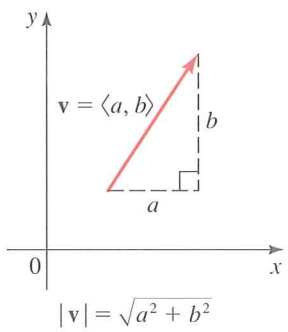


FIGURE 10

MAGNITUDE OF A VERTOR

The **magnitude** or **length** of a vector $\mathbf{v} = \langle a, b\rangle$ is

$$|\mathbf{v}| = \sqrt{a^2 + b^2}$$

EXAMPLE 2 | Magnitudes of Vectors

Find the magnitude of each vector.

(a) $\mathbf{u} = \langle 2, -3\rangle$ **(b)** $\mathbf{v} = \langle 5, 0\rangle$ **(c)** $\mathbf{w} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$

SOLUTION

(a) $|\mathbf{u}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

(b) $|\mathbf{v}| = \sqrt{5^2 + 0^2} = \sqrt{25} = 5$

(c) $|\mathbf{w}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$

NOW TRY EXERCISE **37**

The following definitions of addition, subtraction, and scalar multiplication of vectors correspond to the geometric descriptions given earlier. Figure 11 shows how the analytic definition of addition corresponds to the geometric one.

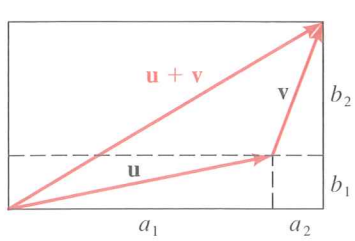


FIGURE 11

ALGEBRAIC OPERATIONS ON VECTORS

If $\mathbf{u} = \langle a_1, b_1\rangle$ and $\mathbf{v} = \langle a_2, b_2\rangle$, then

$$\mathbf{u} + \mathbf{v} = \langle a_1 + a_2, b_1 + b_2\rangle$$

$$\mathbf{u} - \mathbf{v} = \langle a_1 - a_2, b_1 - b_2\rangle$$

$$c\mathbf{u} = \langle ca_1, cb_1\rangle, \qquad c \in \mathbb{R}$$

EXAMPLE 3 | Operations with Vectors

If $\mathbf{u} = \langle 2, -3\rangle$ and $\mathbf{v} = \langle -1, 2\rangle$, find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u}$, $-3\mathbf{v}$, and $2\mathbf{u} + 3\mathbf{v}$.

SOLUTION By the definitions of the vector operations we have

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= \langle 2, -3\rangle - \langle -1, 2\rangle = \langle 1, -1\rangle \\
\mathbf{u} - \mathbf{v} &= \langle 2, -3\rangle - \langle -1, 2\rangle = \langle 3, -5\rangle \\
2\mathbf{u} &= 2\langle 2, -3\rangle = \langle 4, -6\rangle \\
-3\mathbf{v} &= -3\langle -1, 2\rangle = \langle 3, -6\rangle \\
2\mathbf{u} + 3\mathbf{v} &= 2\langle 2, -3\rangle + 3\langle -1, 2\rangle = \langle 4, -6\rangle + \langle -3, 6\rangle = \langle 1, 0\rangle
\end{aligned}$$

NOW TRY EXERCISE **31**

The following properties for vector operations can be easily proved from the definitions. The **zero vector** is the vector $\mathbf{0} = \langle 0, 0\rangle$. It plays the same role for addition of vectors as the number 0 does for addition of real numbers.

PROPERTIES OF VECTORS

Vector addition	Multiplication by a scalar
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
$\mathbf{u} + \mathbf{0} = \mathbf{u}$	$(cd)\mathbf{u} = c(d\mathbf{u}) = d(c\mathbf{u})$
$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	$1\mathbf{u} = \mathbf{u}$
Length of a vector	$0\mathbf{u} = \mathbf{0}$
$\lvert c\mathbf{u}\rvert = \lvert c\rvert\,\lvert\mathbf{u}\rvert$	$c\mathbf{0} = \mathbf{0}$

A vector of length 1 is called a **unit vector**. For instance, in Example 2(c) the vector $\mathbf{w} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$ is a unit vector. Two useful unit vectors are $\mathbf{i}$ and $\mathbf{j}$, defined by

$$\mathbf{i} = \langle 1, 0\rangle \qquad \mathbf{j} = \langle 0, 1\rangle$$

(See Figure 12.) These vectors are special because any vector can be expressed in terms of them. (See Figure 13.)

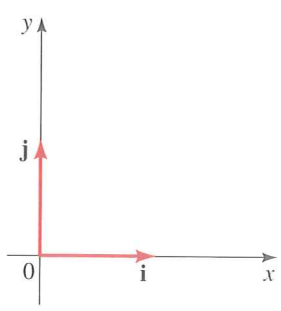


FIGURE 12

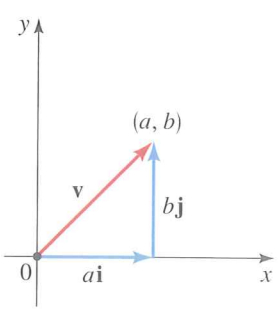


FIGURE 13

VECTORS IN TERMS OF i AND j

The vector $\mathbf{v} = \langle a, b\rangle$ can be expressed in terms of $\mathbf{i}$ and $\mathbf{j}$ by

$$\mathbf{v} = \langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$$

EXAMPLE 4 | Vectors in Terms of i and j

(a) Write the vector $\mathbf{u} = \langle 5, -8\rangle$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(b) If $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 6\mathbf{j}$, write $2\mathbf{u} + 5\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

SOLUTION

(a) $\mathbf{u} = 5\mathbf{i} + (-8)\mathbf{j} = 5\mathbf{i} - 8\mathbf{j}$

(b) The properties of addition and scalar multiplication of vectors show that we can manipulate vectors in the same way as algebraic expressions. Thus

$$\begin{aligned} 2\mathbf{u} + 5\mathbf{v} &= 2(3\mathbf{i} + 2\mathbf{j}) + 5(-\mathbf{i} + 6\mathbf{j}) \\ &= (6\mathbf{i} + 4\mathbf{j}) + (-5\mathbf{i} + 30\mathbf{j}) \\ &= \mathbf{i} + 34\mathbf{j} \end{aligned}$$

NOW TRY EXERCISES **27** AND **35**

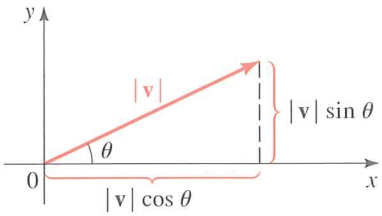


FIGURE 14

Let $\mathbf{v}$ be a vector in the plane with its initial point at the origin. The **direction** of $\mathbf{v}$ is θ, the smallest positive angle in standard position formed by the positive x-axis and $\mathbf{v}$ (see Figure 14). If we know the magnitude and direction of a vector, then Figure 14 shows that we can find the horizontal and vertical components of the vector.

HORIZONTAL AND VERTICAL COMPONENTS OF A VECTOR

Let $\mathbf{v}$ be a vector with magnitude $|\mathbf{v}|$ and direction θ.
Then $\mathbf{v} = \langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$, where

$$a = |\mathbf{v}| \cos\theta \qquad \text{and} \qquad b = |\mathbf{v}| \sin\theta$$

Thus we can express $\mathbf{v}$ as

$$\mathbf{v} = |\mathbf{v}| \cos\theta\,\mathbf{i} + |\mathbf{v}| \sin\theta\,\mathbf{j}$$

EXAMPLE 5 | Components and Direction of a Vector

(a) A vector $\mathbf{v}$ has length 8 and direction $\pi/3$. Find the horizontal and vertical components, and write $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

(b) Find the direction of the vector $\mathbf{u} = -\sqrt{3}\mathbf{i} + \mathbf{j}$.

SOLUTION

(a) We have $\mathbf{v} = \langle a, b\rangle$, where the components are given by

$$a = 8\cos\frac{\pi}{3} = 4 \qquad \text{and} \qquad b = 8\sin\frac{\pi}{3} = 4\sqrt{3}$$

Thus $\mathbf{v} = \langle 4, 4\sqrt{3}\rangle = 4\mathbf{i} + 4\sqrt{3}\mathbf{j}$.

(b) From Figure 15 we see that the direction θ has the property that

$$\tan\theta = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Thus the reference angle for θ is $\pi/6$. Since the terminal point of the vector $\mathbf{u}$ is in Quadrant II, it follows that $\theta = 5\pi/6$.

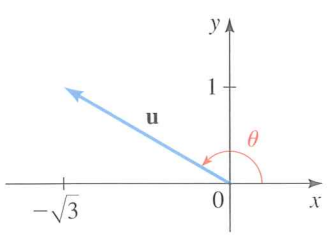


FIGURE 15

NOW TRY EXERCISES **41** AND **51**

Using Vectors to Model Velocity and Force

The **velocity** of a moving object is modeled by a vector whose direction is the direction of motion and whose magnitude is the speed. Figure 16 on the next page shows some vectors $\mathbf{u}$, representing the velocity of wind flowing in the direction N 30° E, and a vector $\mathbf{v}$, representing the velocity of an airplane flying through this wind at the point P. It's obvious from our experience that wind affects both the speed and the direction of an airplane.

The use of bearings (such as N 30° E) to describe directions is explained on page 478 in Section 6.6.

Figure 17 indicates that the true velocity of the plane (relative to the ground) is given by the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$.

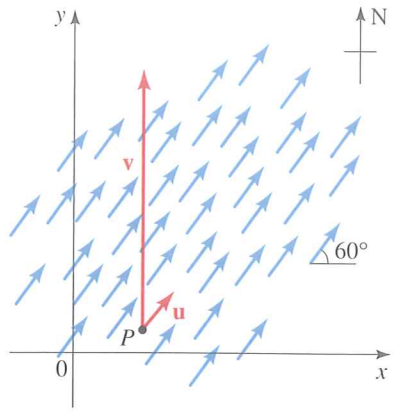


FIGURE 16

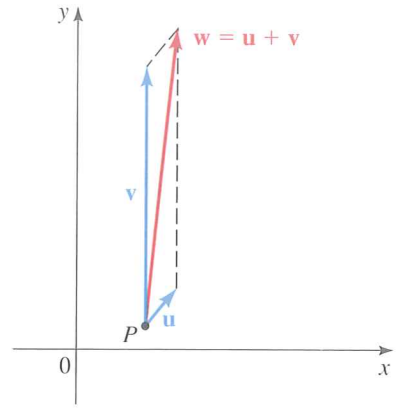


FIGURE 17

EXAMPLE 6 | The True Speed and Direction of an Airplane

An airplane heads due north at 300 mi/h. It experiences a 40 mi/h crosswind flowing in the direction N 30° E, as shown in Figure 16.

(a) Express the velocity $\mathbf{v}$ of the airplane relative to the air, and the velocity $\mathbf{u}$ of the wind, in component form.

(b) Find the true velocity of the airplane as a vector.

(c) Find the true speed and direction of the airplane.

SOLUTION

(a) The velocity of the airplane relative to the air is $\mathbf{v} = 0\mathbf{i} + 300\mathbf{j} = 300\mathbf{j}$. By the formulas for the components of a vector, we find that the velocity of the wind is

$$\begin{aligned}\mathbf{u} &= (40 \cos 60°)\mathbf{i} + (40 \sin 60°)\mathbf{j}\\ &= 20\mathbf{i} + 20\sqrt{3}\mathbf{j}\\ &\approx 20\mathbf{i} + 34.64\mathbf{j}\end{aligned}$$

(b) The true velocity of the airplane is given by the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$:

$$\begin{aligned}\mathbf{w} = \mathbf{u} + \mathbf{v} &= (20\mathbf{i} + 20\sqrt{3}\mathbf{j}) + (300\mathbf{j})\\ &= 20\mathbf{i} + (20\sqrt{3} + 300)\mathbf{j}\\ &\approx 20\mathbf{i} + 334.64\mathbf{j}\end{aligned}$$

(c) The true speed of the airplane is given by the magnitude of $\mathbf{w}$:

$$|\mathbf{w}| \approx \sqrt{(20)^2 + (334.64)^2} \approx 335.2 \text{ mi/h}$$

The direction of the airplane is the direction θ of the vector $\mathbf{w}$. The angle θ has the property that $\tan\theta \approx 334.64/20 = 16.732$, so $\theta \approx 86.6°$. Thus the airplane is heading in the direction N 3.4° E.

NOW TRY EXERCISE **59**

EXAMPLE 7 | Calculating a Heading

A woman launches a boat from one shore of a straight river and wants to land at the point directly on the opposite shore. If the speed of the boat (relative to the water) is 10 mi/h and the river is flowing east at the rate of 5 mi/h, in what direction should she head the boat in order to arrive at the desired landing point?

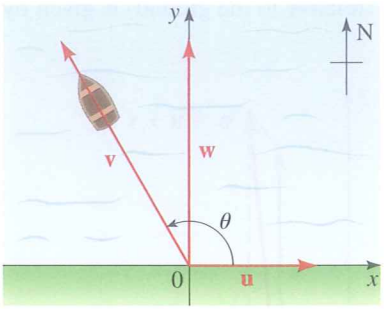


FIGURE 18

SOLUTION We choose a coordinate system with the origin at the initial position of the boat as shown in Figure 18. Let $\mathbf{u}$ and $\mathbf{v}$ represent the velocities of the river and the boat, respectively. Clearly, $\mathbf{u} = 5\mathbf{i}$, and since the speed of the boat is 10 mi/h, we have $|\mathbf{v}| = 10$, so

$$\mathbf{v} = (10 \cos \theta)\mathbf{i} + (10 \sin \theta)\mathbf{j}$$

where the angle θ is as shown in Figure 16. The true course of the boat is given by the vector $\mathbf{w} = \mathbf{u} + \mathbf{v}$. We have

$$\begin{aligned} \mathbf{w} = \mathbf{u} + \mathbf{v} &= 5\mathbf{i} + (10 \cos \theta)\mathbf{i} + (10 \sin \theta)\mathbf{j} \\ &= (5 + 10 \cos \theta)\mathbf{i} + (10 \sin \theta)\mathbf{j} \end{aligned}$$

Since the woman wants to land at a point directly across the river, her direction should have horizontal component 0. In other words, she should choose θ in such a way that

$$\begin{aligned} 5 + 10 \cos \theta &= 0 \\ \cos \theta &= -\tfrac{1}{2} \\ \theta &= 120° \end{aligned}$$

Thus she should head the boat in the direction $\theta = 120°$ (or N 30° W).

NOW TRY EXERCISE **57**

Force is also represented by a vector. Intuitively, we can think of force as describing a push or a pull on an object, for example, a horizontal push of a book across a table or the downward pull of the earth's gravity on a ball. Force is measured in pounds (or in newtons, in the metric system). For instance, a man weighing 200 lb exerts a force of 200 lb downward on the ground. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

EXAMPLE 8 | Resultant Force

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ with magnitudes 10 and 20 lb, respectively, act on an object at a point P as shown in Figure 19. Find the resultant force acting at P.

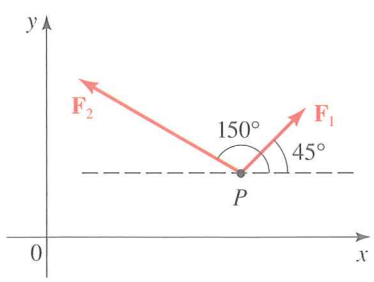


FIGURE 19

SOLUTION We write $\mathbf{F}_1$ and $\mathbf{F}_2$ in component form:

$$\begin{aligned} \mathbf{F}_1 &= (10 \cos 45°)\mathbf{i} + (10 \sin 45°)\mathbf{j} = 10\frac{\sqrt{2}}{2}\mathbf{i} + 10\frac{\sqrt{2}}{2}\mathbf{j} \\ &= 5\sqrt{2}\,\mathbf{i} + 5\sqrt{2}\,\mathbf{j} \\ \mathbf{F}_2 &= (20 \cos 150°)\mathbf{i} + (20 \sin 150°)\mathbf{j} = -20\frac{\sqrt{3}}{2}\mathbf{i} + 20\left(\frac{1}{2}\right)\mathbf{j} \\ &= -10\sqrt{3}\,\mathbf{i} + 10\mathbf{j} \end{aligned}$$

So the resultant force $\mathbf{F}$ is

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 \\ &= (5\sqrt{2}\,\mathbf{i} + 5\sqrt{2}\,\mathbf{j}) + (-10\sqrt{3}\,\mathbf{i} + 10\mathbf{j}) \\ &= (5\sqrt{2} - 10\sqrt{3})\mathbf{i} + (5\sqrt{2} + 10)\mathbf{j} \\ &\approx -10\mathbf{i} + 17\mathbf{j} \end{aligned}$$

The resultant force $\mathbf{F}$ is shown in Figure 20.

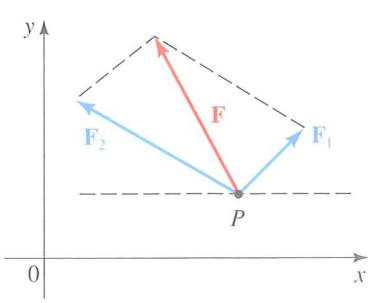


FIGURE 20

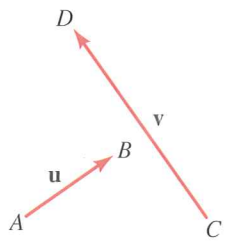
NOW TRY EXERCISE **67**

9.1 EXERCISES

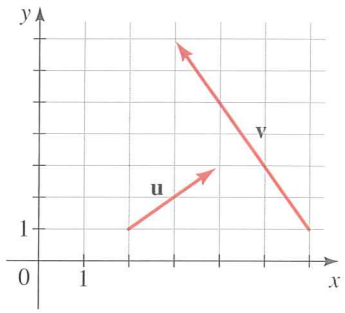
CONCEPTS

1. (a) A vector in the plane is a line segment with an assigned direction. In Figure I below, the vector **u** has initial point _____ and terminal point _____. Sketch the vectors $2\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$.

   (b) A vector in a coordinate plane is expressed by using components. In Figure II below, the vector **u** has initial point (,) and terminal point (,). In component form we write $\mathbf{u} = \langle \quad , \quad \rangle$, and $\mathbf{v} = \langle \quad , \quad \rangle$. Then $2\mathbf{u} = \langle \quad , \quad \rangle$ and $\mathbf{u} + \mathbf{v} = \langle \quad , \quad \rangle$.



I



II

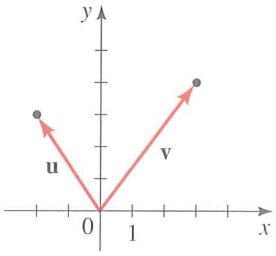
2. (a) The length of a vector $\mathbf{w} = \langle a, b \rangle$ is $|\mathbf{w}| =$ _____, so the length of the vector **u** in Figure II is $|\mathbf{u}| =$ _____.

   (b) If we know the length $|\mathbf{w}|$ and direction θ of a vector **w**, then we can express the vector in component form as $\mathbf{w} = \langle \quad , \quad \rangle$.

SKILLS

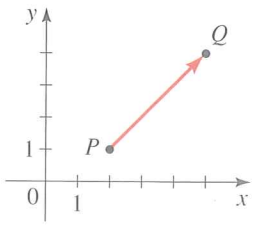
3–8 ■ Sketch the vector indicated. (The vectors **u** and **v** are shown in the figure.)

3. $2\mathbf{u}$
4. $-\mathbf{v}$
5. $\mathbf{u} + \mathbf{v}$
6. $\mathbf{u} - \mathbf{v}$
7. $\mathbf{v} - 2\mathbf{u}$
8. $2\mathbf{u} + \mathbf{v}$



9–18 ■ Express the vector with initial point P and terminal point Q in component form.

9.



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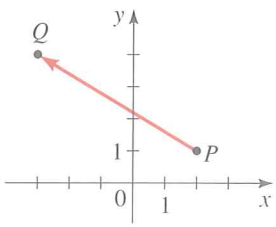

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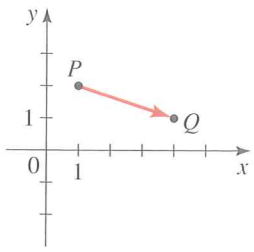
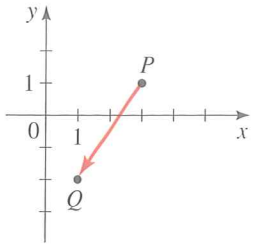


12.



13. $P(3, 2)$, $Q(8, 9)$
14. $P(1, 1)$, $Q(9, 9)$
15. $P(5, 3)$, $Q(1, 0)$
16. $P(-1, 3)$, $Q(-6, -1)$
17. $P(-1, -1)$, $Q(-1, 1)$
18. $P(-8, -6)$, $Q(-1, -1)$

19–22 ■ Sketch the given vector with initial point (4, 3), and find the terminal point.

19. $\mathbf{u} = \langle 2, 4 \rangle$
20. $\mathbf{u} = \langle -1, 2 \rangle$
21. $\mathbf{u} = \langle 4, -3 \rangle$
22. $\mathbf{u} = \langle -8, -1 \rangle$

23–26 ■ Sketch representations of the given vector with initial points at (0, 0), (2, 3), and (−3, 5).

23. $\mathbf{u} = \langle 3, 5 \rangle$
24. $\mathbf{u} = \langle 4, -6 \rangle$
25. $\mathbf{u} = \langle -7, 2 \rangle$
26. $\mathbf{u} = \langle 0, -9 \rangle$

27–30 ■ Write the given vector in terms of **i** and **j**.

27. $\mathbf{u} = \langle 1, 4 \rangle$
28. $\mathbf{u} = \langle -2, 10 \rangle$
29. $\mathbf{u} = \langle 3, 0 \rangle$
30. $\mathbf{u} = \langle 0, -5 \rangle$

31–36 ■ Find $2\mathbf{u}$, $-3\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $3\mathbf{u} - 4\mathbf{v}$ for the given vectors **u** and **v**.

31. $\mathbf{u} = \langle 2, 7 \rangle$, $\mathbf{v} = \langle 3, 1 \rangle$
32. $\mathbf{u} = \langle -2, 5 \rangle$, $\mathbf{v} = \langle 2, -8 \rangle$
33. $\mathbf{u} = \langle 0, -1 \rangle$, $\mathbf{v} = \langle -2, 0 \rangle$
34. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = -2\mathbf{j}$
35. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$
36. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$

37–40 ■ Find $|\mathbf{u}|$, $|\mathbf{v}|$, $|2\mathbf{u}|$, $|\frac{1}{2}\mathbf{v}|$, $|\mathbf{u} + \mathbf{v}|$, $|\mathbf{u} - \mathbf{v}|$, and $|\mathbf{u}| - |\mathbf{v}|$.

37. $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$
38. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$
39. $\mathbf{u} = \langle 10, -1 \rangle$, $\mathbf{v} = \langle -2, -2 \rangle$
40. $\mathbf{u} = \langle -6, 6 \rangle$, $\mathbf{v} = \langle -2, -1 \rangle$

41–46 ■ Find the horizontal and vertical components of the vector with given length and direction, and write the vector in terms of the vectors **i** and **j**.

41. $|\mathbf{v}| = 40$, $\theta = 30°$
42. $|\mathbf{v}| = 50$, $\theta = 120°$
43. $|\mathbf{v}| = 1$, $\theta = 225°$
44. $|\mathbf{v}| = 800$, $\theta = 125°$
45. $|\mathbf{v}| = 4$, $\theta = 10°$
46. $|\mathbf{v}| = \sqrt{3}$, $\theta = 300°$

47–52 ■ Find the magnitude and direction (in degrees) of the vector.

47. $\mathbf{v} = \langle 3, 4 \rangle$

48. $\mathbf{v} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$

49. $\mathbf{v} = \langle -12, 5 \rangle$

50. $\mathbf{v} = \langle 40, 9 \rangle$

51. $\mathbf{v} = \mathbf{i} + \sqrt{3}\,\mathbf{j}$

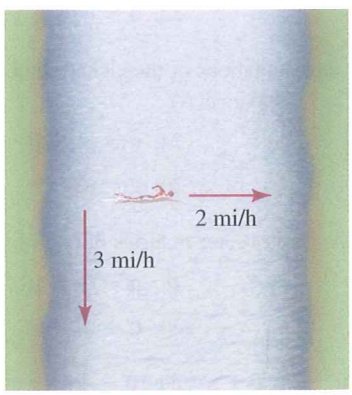
52. $\mathbf{v} = \mathbf{i} + \mathbf{j}$

APPLICATIONS

53. Components of a Force A man pushes a lawn mower with a force of 30 lb exerted at an angle of 30° to the ground. Find the horizontal and vertical components of the force.

54. Components of a Velocity A jet is flying in a direction N 20° E with a speed of 500 mi/h. Find the north and east components of the velocity.

55. Velocity A river flows due south at 3 mi/h. A swimmer attempting to cross the river heads due east swimming at 2 mi/h relative to the water. Find the true velocity of the swimmer as a vector.



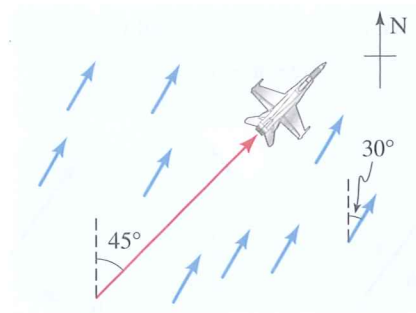
56. Velocity Suppose that in Exercise 55 the current is flowing at 1.2 mi/h due south. In what direction should the swimmer head in order to arrive at a landing point due east of his starting point?

57. Velocity The speed of an airplane is 300 mi/h relative to the air. The wind is blowing due north with a speed of 30 mi/h. In what direction should the airplane head in order to arrive at a point due west of its location?

58. Velocity A migrating salmon heads in the direction N 45° E, swimming at 5 mi/h relative to the water. The prevailing ocean currents flow due east at 3 mi/h. Find the true velocity of the fish as a vector.

59. True Velocity of a Jet A pilot heads his jet due east. The jet has a speed of 425 mi/h relative to the air. The wind is blowing due north with a speed of 40 mi/h.
(a) Express the velocity of the wind as a vector in component form.
(b) Express the velocity of the jet relative to the air as a vector in component form.
(c) Find the true velocity of the jet as a vector.
(d) Find the true speed and direction of the jet.

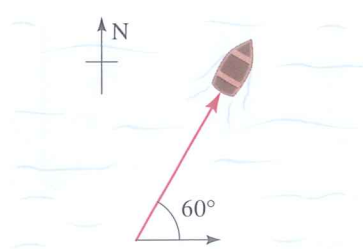
60. True Velocity of a Jet A jet is flying through a wind that is blowing with a speed of 55 mi/h in the direction N 30° E (see the figure). The jet has a speed of 765 mi/h relative to the air, and the pilot heads the jet in the direction N 45° E.
(a) Express the velocity of the wind as a vector in component form.
(b) Express the velocity of the jet relative to the air as a vector in component form.
(c) Find the true velocity of the jet as a vector.
(d) Find the true speed and direction of the jet.



61. True Velocity of a Jet Find the true speed and direction of the jet in Exercise 60 if the pilot heads the plane in the direction N 30° W.

62. True Velocity of a Jet In what direction should the pilot in Exercise 60 head the plane for the true course to be due north?

63. Velocity of a Boat A straight river flows east at a speed of 10 mi/h. A boater starts at the south shore of the river and heads in a direction 60° from the shore (see the figure). The motorboat has a speed of 20 mi/h relative to the water.
(a) Express the velocity of the river as a vector in component form.
(b) Express the velocity of the motorboat relative to the water as a vector in component form.
(c) Find the true velocity of the motorboat.
(d) Find the true speed and direction of the motorboat.



64. Velocity of a Boat The boater in Exercise 63 wants to arrive at a point on the north shore of the river directly opposite the starting point. In what direction should the boat be headed?

65. Velocity of a Boat A boat heads in the direction N 72° E. The speed of the boat relative to the water is 24 mi/h. The water

is flowing directly south. It is observed that the true direction of the boat is directly east.

(a) Express the velocity of the boat relative to the water as a vector in component form.

(b) Find the speed of the water and the true speed of the boat.

66. Velocity A woman walks due west on the deck of an ocean liner at 2 mi/h. The ocean liner is moving due north at a speed of 25 mi/h. Find the speed and direction of the woman relative to the surface of the water.

67–72 ■ Equilibrium of Forces The forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at the same point P are said to be in equilibrium if the resultant force is zero, that is, if $\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = 0$. Find **(a)** the resultant forces acting at P, and **(b)** the additional force required (if any) for the forces to be in equilibrium.

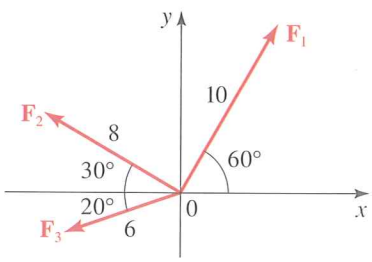
67. $\mathbf{F}_1 = \langle 2, 5\rangle, \quad \mathbf{F}_2 = \langle 3, -8\rangle$

68. $\mathbf{F}_1 = \langle 3, -7\rangle, \quad \mathbf{F}_2 = \langle 4, -2\rangle, \quad \mathbf{F}_3 = \langle -7, 9\rangle$

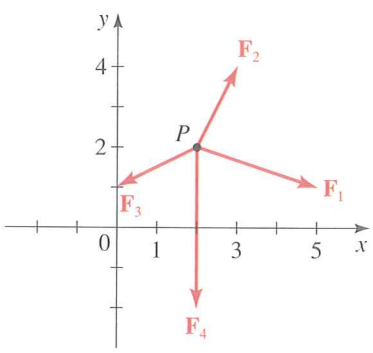
69. $\mathbf{F}_1 = 4\mathbf{i} - \mathbf{j}, \quad \mathbf{F}_2 = 3\mathbf{i} - 7\mathbf{j}, \quad \mathbf{F}_3 = -8\mathbf{i} + 3\mathbf{j}, \quad \mathbf{F}_4 = \mathbf{i} + \mathbf{j}$

70. $\mathbf{F}_1 = \mathbf{i} - \mathbf{j}, \quad \mathbf{F}_2 = \mathbf{i} + \mathbf{j}, \quad \mathbf{F}_3 = -2\mathbf{i} + \mathbf{j}$

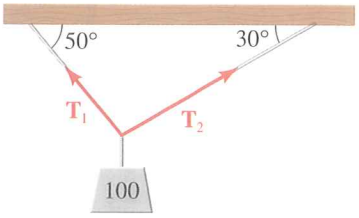
71.



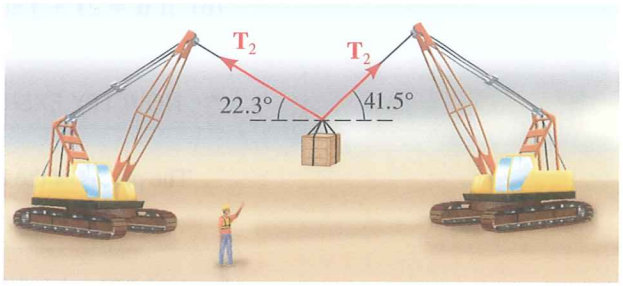
72.



73. Equilibrium of Tensions A 100-lb weight hangs from a string as shown in the figure. Find the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ in the string.

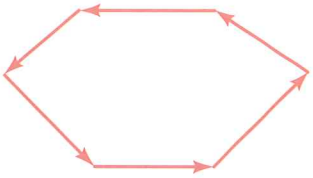


74. Equilibrium of Tensions The cranes in the figure are lifting an object that weighs 18,278 lb. Find the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$.



DISCOVERY ■ DISCUSSION ■ WRITING

75. Vectors That Form a Polygon Suppose that n vectors can be placed head to tail in the plane so that they form a polygon. (The figure shows the case of a hexagon.) Explain why the sum of these vectors is $\mathbf{0}$.



9.2 THE DOT PRODUCT

The Dot Product of Vectors ▶ The Component of **u** Along **v** ▶ The Projection of **u** Onto **v** ▶ Work

In this section we define an operation on vectors called the dot product. This concept is especially useful in calculus and in applications of vectors to physics and engineering.

▼ The Dot Product of Vectors

We begin by defining the dot product of two vectors.

DEFINITION OF THE DOT PRODUCT

If $\mathbf{u} = \langle a_1, b_1 \rangle$ and $\mathbf{v} = \langle a_2, b_2 \rangle$ are vectors, then their **dot product**, denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined by

$$\mathbf{u} \cdot \mathbf{v} = a_1 a_2 + b_1 b_2$$

Thus to find the dot product of **u** and **v**, we multiply corresponding components and add. The dot product is *not* a vector; it is a real number, or scalar.

EXAMPLE 1 | Calculating Dot Products

(a) If $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 5 \rangle$ then

$$\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(5) = 2$$

(b) If $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 6\mathbf{j}$, then

$$\mathbf{u} \cdot \mathbf{v} = (2)(5) + (1)(-6) = 4$$

NOW TRY EXERCISES **5(a)** AND **11(a)**

The proofs of the following properties of the dot product follow easily from the definition.

PROPERTIES OF THE DOT PRODUCT

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (a\mathbf{v})$
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
4. $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$

PROOF We prove only the last property. The proofs of the others are left as exercises. Let $\mathbf{u} = \langle a, b \rangle$. Then

$$\mathbf{u} \cdot \mathbf{u} = \langle a, b \rangle \cdot \langle a, b \rangle = a^2 + b^2 = |\mathbf{u}|^2$$

Let **u** and **v** be vectors, and sketch them with initial points at the origin. We define the **angle θ between u and v** to be the smaller of the angles formed by these representations of **u** and **v** (see Figure 1). Thus $0 \le \theta \le \pi$. The next theorem relates the angle between two vectors to their dot product.

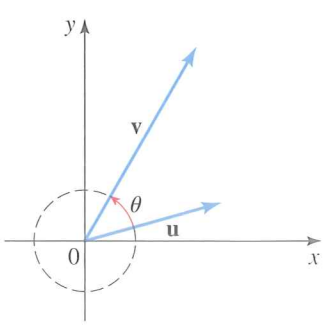


FIGURE 1

THE DOT PRODUCT THEOREM

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$$

PROOF Applying the Law of Cosines to triangle AOB in Figure 2 gives

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos \theta$$