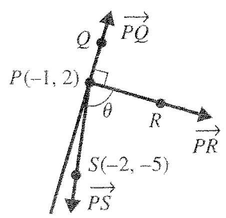
We can determine whether θ is acute or obtuse by calculating the dot product $\overrightarrow{PS} \cdot \overrightarrow{PR}$. If the dot product is positive, then

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{PS} \cdot \overrightarrow{PR}}{\|\overrightarrow{PS}\| \, \|\overrightarrow{PR}\|}\right)$$

is acute and S should be blue. If the dot product is negative, then θ is obtuse and S should be yellow.

Let P be $(-1, 2)$, S be $(-2, -5)$, $\overrightarrow{PQ} = \mathbf{i} + 5\mathbf{j}$, and $\overrightarrow{PR} = 5\mathbf{i} - \mathbf{j}$. Determine the appropriate color at point S. See the figure below.

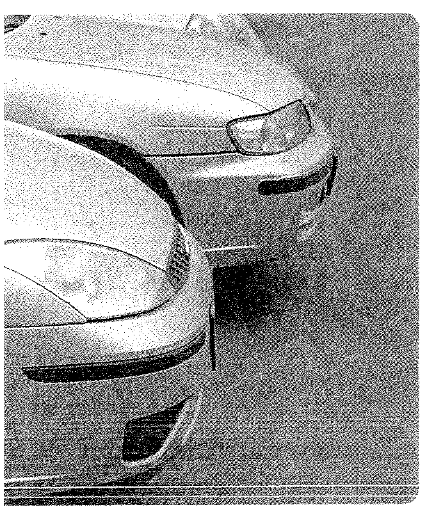


2. **Dot Products in Computer Graphics** A computer screen is gray to the left of vector $\overrightarrow{PQ} = 2\mathbf{i} - 3\mathbf{j}$ and blue to the right, where vector $\overrightarrow{PR} = 3\mathbf{i} + 2\mathbf{j}$ is perpendicular to $\overrightarrow{PQ}$. Let point P be $(2, 1)$. Determine the color of a pixel located at S.
   (a) $S = (3, -2)$ **(b)** $S = (2, 2)$

3. **Dot Products in Computer Graphics** A computer screen is to be blue above vector $\overrightarrow{PQ} = 5\mathbf{i} - \mathbf{j}$ and white below, where point P is $(2, 1)$. Determine the color of a pixel located at S.
   (a) $S = (100, -10)$ **(b)** $S = (-500, 50)$

8.4 Parametric Equations

- **Learn basic concepts about parametric equations**
- **Graph parametric equations**
- **Use parametric equations to solve applications**



Introduction

Sometimes a curve cannot be modeled by a function. For example, a circle cannot be described by a single function because a circle fails the vertical line test. *Parametric equations* represent a different approach to describing curves in the xy-plane. They are used in industry to draw complicated curves and surfaces, such as the hood of an automobile. Parametric equations are also used in computer graphics, engineering, and physics. (***Source:*** F. Hill, *Computer Graphics*.)

Basic Concepts

Some curves cannot be represented by $y = f(x)$, but they can be represented by parametric equations. See Figures 8.66–8.68.

Curves That Are Not Functions

[−6, 6, 1] by [−4, 4, 1]

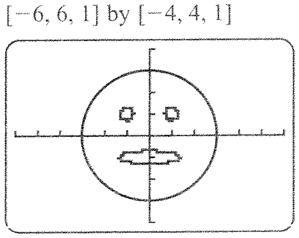


Figure 8.66

[−6, 6, 1] by [−4, 4, 1]

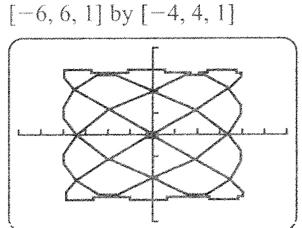


Figure 8.67

[−6, 6, 1] by [−4, 4, 1]

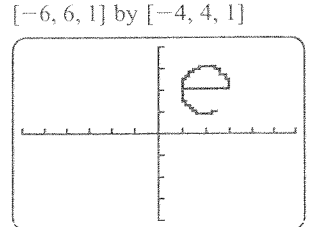


Figure 8.68

We now define parametric equations of a plane curve.

PARAMETRIC EQUATIONS OF A PLANE CURVE

A **plane curve** is a set of points (x, y) such that $x = f(t)$ and $y = g(t)$, where f and g are continuous functions on an interval $a \leq t \leq b$. The equations $x = f(t)$ and $y = g(t)$ are **parametric equations** with **parameter** t.

Parametric equations can be represented symbolically, numerically, graphically, and verbally. This is illustrated in the next example.

EXAMPLE 1 Representing parametric equations

Let $x = t + 3$ and $y = t^2$ for $-3 \leq t \leq 3$.

(a) Make a table of values for x and y with $t = -3, -2, -1, \ldots, 3$.

(b) Plot the points in the table and graph the curve. Add arrows to show how the curve is traced out.

(c) Describe the curve.

SOLUTION

(a) *Numerical Representation* A numerical representation of the parametric equations is shown in Table 8.1. For example, if $t = 2$, then $x = 2 + 3 = 5$ and $y = 2^2 = 4$.

Points (x, y) on a Plane Curve

Parameter t	t	-3	-2	-1	0	1	2	3
$x = t + 3$	x	0	1	2	3	4	5	6
$y = t^2$	y	9	4	1	0	1	4	9

Table 8.1

(b) *Graphical Representation* Each ordered pair (x, y) in Table 8.1 is plotted in Figure 8.69, and then the points are connected to obtain the curve.

Curve Defined Parametrically

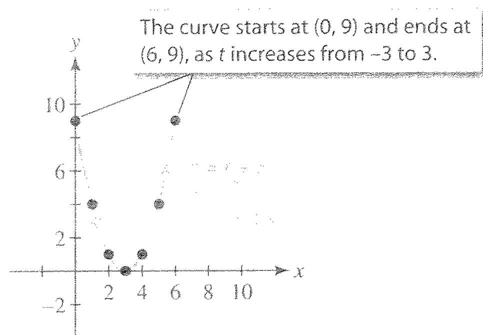


Figure 8.69

(c) *Verbal Representation* The curve in Figure 8.69 appears to be the lower portion of a parabola with vertex (3, 0). See Example 2.

Now Try Exercise 1

Graphing Calculators (Optional) Graphing calculators are capable of using parametric equations to make tables and graphs. In addition to setting values for the viewing rectangle, we must specify the interval for t. A window setting, table, and graph for the parametric equations in Example 1 are shown in Figures 8.70–8.72. The variable Tstep represents the increment in the parameter t and has a value of 0.1 in this case.

Calculator Help
To create the graph shown in Figure 8.72 with parametric equations, see Appendix A (page AP-18).

Setting a Window

```
WINDOW
  Tmin=⁻3
  Tmax=3
  Tstep=.1
  Xmin=⁻2
  Xmax=10
  Xscl=1
↓ Ymin=⁻2
```

Figure 8.70

Table of Points

T	X1T	Y1T
⁻3	0	9
⁻2	1	4
⁻1	2	1
0	3	0
1	4	1
2	5	4
3	6	9

T=⁻3

Figure 8.71

Plane Curve

$[-2, 10, 1]$ by $[-2, 10, 1]$

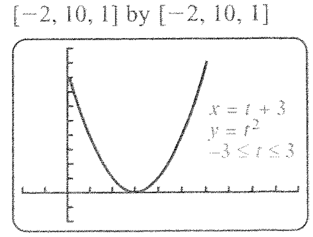


Figure 8.72

Converting and Graphing Equations

We can verify symbolically that the curve in Figure 8.69 in Example 1 is indeed a portion of a parabola, as demonstrated in the next example.

EXAMPLE 2 Finding an equivalent rectangular equation

Find an equivalent rectangular equation for $x = t + 3$ and $y = t^2$, where $-3 \le t \le 3$. Note that these parametric equations were discussed in Example 1.

SOLUTION Begin by solving $x = t + 3$ for t to obtain $t = x - 3$. Substituting for t in $y = t^2$ results in

$$y = (x - 3)^2,$$

which represents a parabola with vertex $(3, 0)$. When $t = -3$ then $x = 0$, and when $t = 3$ then $x = 6$. Thus the domain is restricted to $0 \le x \le 6$. See Figure 8.69.

Now Try Exercise 15

EXAMPLE 3 Finding equivalent rectangular equations

Find an equivalent rectangular equation for each pair of parametric equations. Use the rectangular equation to help graph the parametric equation. Add arrows to show how the curve is traced out.

(a) $x = 4t,\ y = t - 3;\ -\infty < t < \infty$

(b) $x = \sqrt{4 - t^2},\ y = t;\ -2 \le t \le 2$

A Line

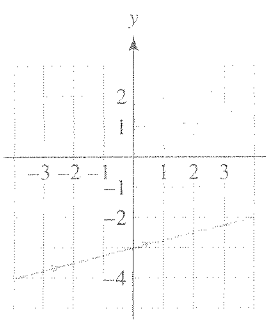


Figure 8.73

SOLUTION

(a) Start by solving $x = 4t$ for t to obtain $t = \frac{1}{4}x$. Substitute for t in the given parametric equation for y.

$$y = t - 3$$

$$y = \frac{1}{4}x - 3$$

Because $y = \frac{1}{4}x - 3$, these parametric equations trace out a line with slope $\frac{1}{4}$ and y-intercept -3. As t increases, x also increases, so this line is traced out from left to right, as illustrated in Figure 8.73. Note that t can be any real number.

(b) Because $y = t$, it follows that a rectangular equation is $x = \sqrt{4 - y^2}$. To determine the graph of this equation, square each side.

$$x = \sqrt{4 - y^2} \qquad \text{Rectangular equation}$$

$$x^2 = 4 - y^2 \qquad \text{Square each side.}$$

$$x^2 + y^2 = 4 \qquad \text{Add } y^2 \text{ to each side.}$$

The equation $x^2 + y^2 = 4$ is a circle with center $(0, 0)$ and radius 2. Because $\sqrt{4 - y^2}$ is never negative, it follows that $x \ge 0$. Thus the parametric equation traces out only the right half of this circle. See Figure 8.74. Because $y = t$, this semicircle is traced from bottom to top as y increases from -2 to 2.

Now Try Exercises 9 and 13

A Semicircle

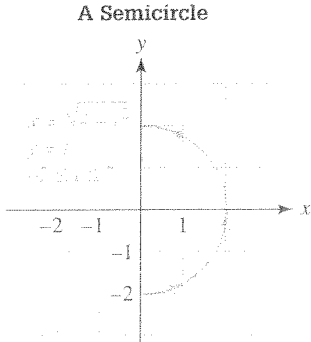


Figure 8.74

Parametric equations can model a circle, as shown in the next example.

EXAMPLE 4 Graphing a circle with parametric equations

Graph $x = 2 \cos t$ and $y = 2 \sin t$ for $0 \le t \le 2\pi$. Find an equivalent equation by using rectangular coordinates.

SOLUTION Enter and graph these parametric equations, as shown in Figures 8.75 and 8.76. Be sure to have the mode of the calculator set for parametric equations. (Note that Tstep = 0.1.)

Enter Parametric Equations

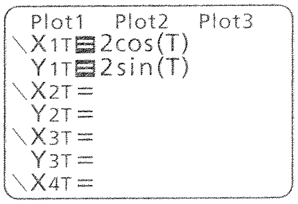


Figure 8.75

Graph Equations to Form a Circle ($r = 2$)

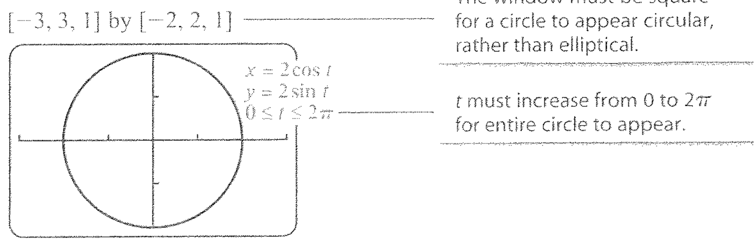


Figure 8.76

To verify that this is a circle, consider the following.

$$
\begin{aligned}
{}^2 + y^2 &= (2\cos t)^2 + (2\sin t)^2 \\
&= 4\cos^2 t + 4\sin^2 t \\
&= 4(\cos^2 t + \sin^2 t) \\
&= 4
\end{aligned}
$$

The parametric equations are equivalent to $x^2 + y^2 = 4$, which is a circle with its center at $(0, 0)$ and having radius 2.

Now Try Exercise 19

In the next two examples, an equation written in terms of x and y is converted to parametric equations.

EXAMPLE 5 Converting to parametric equations

Convert $x = y^2 - 4y + 4$ to parametric equations.

SOLUTION There is more than one way to convert this equation to parametric equations. One simple way is to let $y = t$ and then write the parametric equations as

$$x = t^2 - 4t + 4, \quad y = t,$$

where t is any real number. To write a different pair of parametric equations, note that

$$x = y^2 - 4y + 4 = (y - 2)^2.$$

Let $t = y - 2$, or $y = t + 2$, and then another pair of parametric equations is

$$x = t^2, y = t + 2.$$

Now Try Exercise 53

EXAMPLE 6 Converting to parametric equations

Given the equation $x^2 + y^2 = 1$, complete the following.

(a) Find parametric equations for this equation.

(b) What portion of the graph appears for $0 \le t \le \pi$?

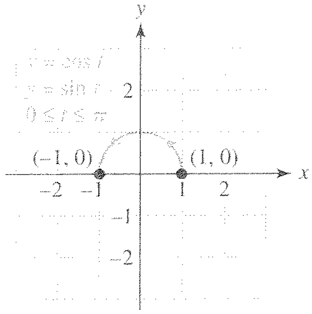


Figure 8.77

SOLUTION

(a) The graph of $x^2 + y^2 = 1$ is the unit circle. From trigonometry we know that on the unit circle $x = \cos t$ and $y = \sin t$. Since $\cos^2 t + \sin^2 t = 1$ for all t, we have the following result.

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

Thus parametric equations for the unit circle are

$$x = \cos t, \qquad y = \sin t; \qquad 0 \le t \le 2\pi.$$

(b) When t increases from 0 to π, the upper half of the circle is graphed, moving from the point (1, 0) to the point (−1, 0). See Figure 8.77.

Now Try Exercise 49

NOTE The equations $x = a\cos t$, $y = a\sin t$ for $0 \le t \le 2\pi$ trace out a circle with radius $r = a$. If t is limited to an interval that is less than 2π in length, then only a portion of a circle will appear.

Applications of Parametric Equations

Parametric equations are used to simulate motion. If a ball or shot is thrown with a velocity of v feet per second at an angle θ with the horizontal, its flight can be modeled by the parametric equations

$$x = (v\cos\theta)t \quad \text{and} \quad y = (v\sin\theta)t - 16t^2 + h,$$

where t is in seconds and h is the initial height above the ground. The term $-16t^2$ occurs because gravity is pulling downward. See Figure 8.78. (These equations ignore air resistance.)

Modeling the Path of a Shot

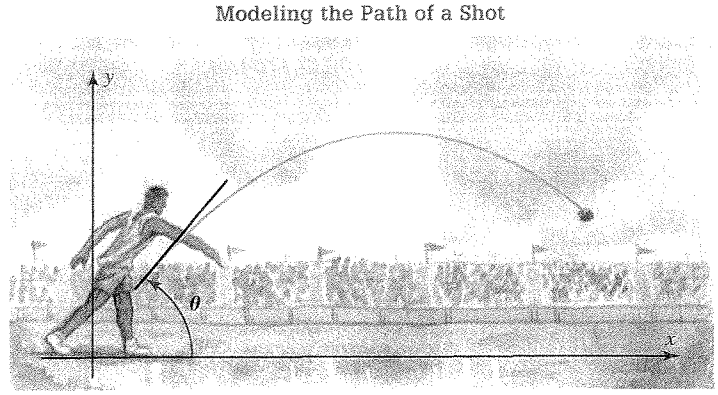


Figure 8.78

EXAMPLE 7 **Simulating motion with parametric equations**

Three golf balls are hit simultaneously into the air at 132 feet per second (90 miles per hour), making angles of 30°, 50°, and 70° with the horizontal.

(a) Assuming the ground is level, determine graphically which ball travels the farthest horizontally. Estimate this distance.

(b) Which ball reaches the greatest height? Estimate this height.

Golf Ball Hit at Three Angles

[0, 600, 50] by [0, 400, 50]

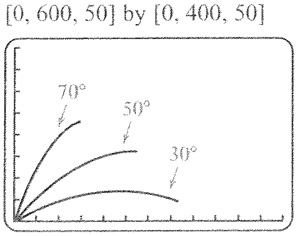


Figure 8.79 Degree Mode

SOLUTION

(a) The three sets of parametric equations determined by the three golf balls are as follows. Since $h = 0$, the only difference between the three balls is the angle of elevation.

$X_1 = 132\cos(30)T$, $\quad Y_1 = 132\sin(30)T - 16T\text{^}2$ — Hit at 30°

$X_2 = 132\cos(50)T$, $\quad Y_2 = 132\sin(50)T - 16T\text{^}2$ — Hit at 50°

$X_3 = 132\cos(70)T$, $\quad Y_3 = 132\sin(70)T - 16T\text{^}2$ — Hit at 70°

Angles Affect the Distance
[0, 600, 50] by [0, 400, 50]

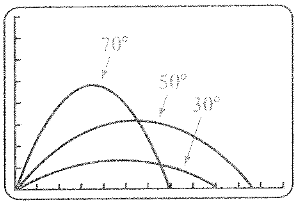


Figure 8.80 Degree Mode

The graphs of the three sets of parametric equations are shown in Figures 8.79 and 8.80, where $0 \le t \le 9$. A graphing calculator in *simultaneous mode* has been used so that we can view all three balls in flight at the same time. From the second graph we can see that the ball hit at 50° travels the farthest distance. Using the trace feature, we estimate this horizontal distance to be about 540 feet.

(b) Using the trace feature, the ball hit at 70° reaches the greatest height of about 240 feet.

Now Try Exercise 71

CLASS DISCUSSION

If a golf ball is hit at 88 feet per second (60 mi/hr), use trial and error to find the angle θ that results in a maximum distance for the ball.

EXAMPLE 8 **Modeling the flight of a baseball**

A baseball is hit from a height of 4 feet at a 30° angle above the horizontal. Its initial velocity is 128 feet per second. See Figure 8.81.

(a) Write parametric equations that model the flight of the baseball.

(b) Determine the horizontal distance that the ball travels in the air, assuming that the ground is level.

(c) What is the maximum height of the baseball?

(d) Would the ball clear a 4-foot-high fence that is 400 feet from the batter?

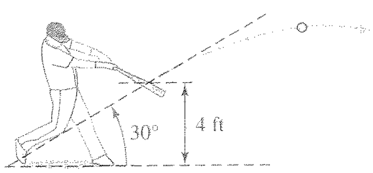


Figure 8.81

SOLUTION

(a) Let $v = 128$, $\theta = 30°$, and $h = 4$. Then the parametric equations become

$$x = (128 \cos 30°)t \qquad \text{and} \qquad y = (128 \sin 30°)t - 16t^2 + 4.$$

Since $\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 30° = \frac{1}{2}$, these equations can be rewritten as

$$x = (64\sqrt{3})t \qquad \text{and} \qquad y = 64t - 16t^2 + 4.$$

(b) To find how far the ball travels, we first determine the length of time that the ball is in flight. The ball hits the ground when $y = 0$.

$$64t - 16t^2 + 4 = 0$$
$$16t^2 - 64t - 4 = 0$$
$$t = \frac{64 \pm \sqrt{(-64)^2 - 4(16)(-4)}}{2(16)}$$
$$t \approx 4.0616 \qquad \text{or} \qquad t \approx -0.0616$$

After 4.0616 seconds, the ball traveled *horizontally* $x = 64\sqrt{3}(4.0616) \approx 450.2$ feet.

(c) The graph of $y = 64t - 16t^2 + 4$ is a parabola that opens downward. Using the *vertex formula*, we find that the maximum height of the ball occurs after

$$t = -\frac{b}{2a} = -\frac{64}{2(-16)} = 2 \text{ seconds.}$$

The maximum height is $y = 64(2) - 16(2)^2 + 4 = 68$ feet.

(d) Because $x = (64\sqrt{3})t$, we can determine how long it takes the ball to reach the fence by solving the equation $(64\sqrt{3})t = 400$.

$$(64\sqrt{3})t = 400, \qquad \text{or} \qquad t = \frac{400}{64\sqrt{3}} \approx 3.61 \text{ seconds}$$

After 3.61 seconds, the ball has traveled horizontally 400 feet and is

$$y = 64(3.61) - 16(3.61)^2 + 4 \approx 27 \text{ feet}$$

high. The baseball easily clears the 4-foot fence.

Now Try Exercise 75

An Application from Computer Graphics Parametric equations are used frequently in computer graphics to design a variety of figures and letters. Computer fonts are sometimes designed using parametric equations. In the next example, we use parametric equations to design a "smiley" face consisting of a head, two eyes, and a mouth. (***Source:*** F. Hill, *Computer Graphics*.)

EXAMPLE 9 **Creating drawings with parametric equations**

Graph a "smiley" face using parametric equations. Answers may vary.

SOLUTION

Head We can use a circle centered at the origin for the head. If the radius is 2, then let $x = 2\cos t$ and $y = 2\sin t$ for $0 \le t \le 2\pi$. These equations are graphed in Figure 8.82.

Eyes For the eyes we can use two small circles. The eye in the first quadrant can be modeled by $x = 1 + 0.3\cos t$ and $y = 1 + 0.3\sin t$ for $0 \le t \le 2\pi$. This represents a circle centered at $(1, 1)$ with radius 0.3. The eye in the second quadrant can be modeled by $x = -1 + 0.3\cos t$ and $y = 1 + 0.3\sin t$ for $0 \le t \le 2\pi$, which is a circle centered at $(-1, 1)$ with radius 0.3. These equations are graphed in Figure 8.83.

Start with the Head
$[-3, 3, 1]$ by $[-2, 2, 1]$

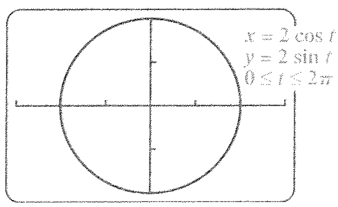


Figure 8.82

Add Eyes
$[-3, 3, 1]$ by $[-2, 2, 1]$

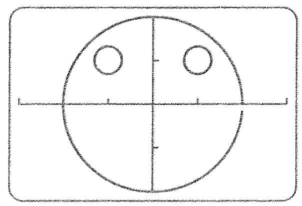


Figure 8.83

Add Mouth and Pupils
$[-3, 3, 1]$ by $[-2, 2, 1]$

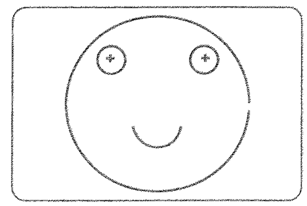


Figure 8.84

CLASS DISCUSSION

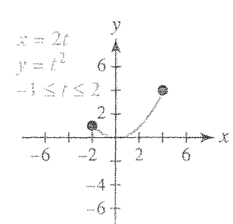
Modify the face in Example 9 so that it is frowning. Try to find a way to make the right eye shut rather than open.

Mouth For the smile we can use the lower half of a circle. Using trial and error, we might arrive at $x = 0.5\cos \frac{1}{2}t$ and $y = -0.5 - 0.5\sin \frac{1}{2}t$. This is a semicircle centered at $(0, -0.5)$ with radius 0.5. Since we are letting $0 \le t \le 2\pi$, the term $\frac{1}{2}t$ ensures that only half a circle (a semicircle) is drawn. The minus sign before $0.5\sin \frac{1}{2}t$ in the y-equation causes the lower half of the semicircle to be drawn rather than the upper half. The final result is shown in Figure 8.84. The pupils have been added by plotting the points $(1, 1)$ and $(-1, 1)$, and the coordinate axes have been turned off.

Now Try Exercise 69

8.4 Putting It All Together

Parametric equations can be used to model a wide variety of curves in the xy-plane that cannot be represented by a single function.

CONCEPT	EXPLANATION	EXAMPLES
Plane curve and parametric equations	A plane curve is a set of points (x, y) such that $x = f(t)$ and $y = g(t)$, where f and g are continuous on an interval $a \le t \le b$. The equations $x = f(t)$ and $y = g(t)$ are parametric equations with parameter t.	If $x = 2t$ and $y = t^2$ for $-1 \le t \le 2$, then the resulting graph is a portion of a parabola. 

CONCEPT	EXPLANATION	EXAMPLES
Writing parametric equations in terms of x and y	Solve one of the parametric equations for t and substitute into the second equation.	If $x = t^3$ and $y = t^2 - 2$, solve $x = t^3$ for t to obtain $t = x^{1/3}$. Substituting gives $y = x^{2/3} - 2$.
Converting to parametric equations	If possible, solve the equation for one of the variables. Let the other variable equal t. Now write the first variable in terms of t. Answers may vary.	If $4x = y^2$, solve the equation for x to obtain $x = \frac{1}{4}y^2$. Let $y = t$. Then $x = \frac{1}{4}t^2$ and $y = t$. Another possibility is $x = t^2$ and $y = 2t$.

8.4 Exercises

Graphs of Parametric Equations

Exercises 1–8: Use the parametric equations to complete the following.

(a) Make a table of values for $t = 0, 1, 2, 3$.

(b) Plot the points from the table and graph the curve for $0 \le t \le 3$. Add arrows to show how the curve is traced out.

(c) Describe the curve.

1. $x = t - 1,\quad y = 2t$
2. $x = t + 1,\quad y = t - 2$
3. $x = t + 2,\quad y = (t - 2)^2$
4. $x = \frac{1}{3}t^2,\quad y = t - 1$
5. $x = \sqrt{9 - t^2},\quad y = t$
6. $x = t^2,\quad y = 2t + 1$
7. $x = t,\quad y = \sqrt{9 - t^2}$
8. $x = 3t,\quad y = t^2 + 2$

Exercises 9–24: Find a rectangular equation for each curve and describe the curve. Support your result by graphing the parametric equations.

9. $x = 3t,\quad y = t - 1;\quad -\infty < t < \infty$
10. $x = t + 3,\quad y = 2t;\quad -\infty < t < \infty$
11. $x = 3t^2,\quad y = t + 1;\quad -\infty < t < \infty$
12. $x = t^2 - 2t + 1,\quad y = t - 1;\quad -\infty < t < \infty$
13. $x = \sqrt{1 - t^2},\quad y = t;\quad -1 \le t \le 1$
14. $x = t,\quad y = \sqrt{9 - t^2};\quad -3 \le t \le 3$
15. $x = t,\quad y = \frac{1}{2}t^2;\quad -2 \le t \le 2$
16. $x = \sqrt[3]{t},\quad y = t;\quad -2 \le t \le 2$
17. $x = t - 2,\quad y = t^2 + 1;\quad -1 \le t \le 2$
18. $x = 2t,\quad y = t^2 + 1;\quad -1 \le t \le 2$
19. $x = 3 \sin t,\quad y = 3 \cos t;\quad -\pi \le t \le \pi$
20. $x = 4 \cos t,\quad y = 4 \sin t;\quad 0 \le t \le 2\pi$
21. $x = 2 \sin t,\quad y = -2 \cos t;\quad 0 \le t \le 2\pi$
22. $x = \cos 2t,\quad y = \sin 2t;\quad 0 \le t \le \pi$
23. $x = 3 \cos 2t,\quad y = 3 \sin 2t;\quad 0 \le t \le \pi$
24. $x = 2 \cos^2 t,\quad y = 2 \sin^2 t;\quad 0 \le t \le \frac{\pi}{2}$

Exercises 25–42: Graph the parametric equations.

25. $x = \frac{1}{3}t,\quad y = \frac{2}{3}t + 1;\quad -\infty < t < \infty$
26. $x = t + 3,\quad y = 2t - 1;\quad -\infty < t < \infty$
27. $x = t^2,\quad y = 2t;\quad -\infty < t < \infty$
28. $x = \frac{1}{2}(t + 2)^2,\quad y = t + 2;\quad -\infty < t < \infty$
29. $x = \cos t,\quad y = \sin t;\quad 0 \le t \le \pi$
30. $x = 2 \sin t,\quad y = 2 \cos t;\quad -\pi \le t \le 0$

31. $x = t^3, \quad y = t^2; \quad -2 \le t \le 2$

32. $x = e^t, \quad y = t - 1; \quad -2 \le t \le 2$

33. $x = t^2, \quad y = \ln t; \quad 0 < t \le 2$

34. $x = t^3 - t, \quad y = e^t; \quad -1.5 \le t \le 1.5$

35. $x = t - \sin t, \quad y = 1 - \cos t; \quad 0 \le t \le 6\pi$

36. $x = t^3 + 3t, \quad y = 2\cos t; \quad -1 \le t \le 1$

37. $x = 2 + \cos t, \quad y = \sin t - 1; \quad 0 \le t \le 2\pi$

38. $x = -2 + \cos t, \quad y = \sin t + 1; \quad 0 \le t \le 2\pi$

39. $x = \cos^3 t, \quad y = \sin^3 t; \quad 0 \le t \le 2\pi$

40. $x = \cos^5 t, \quad y = \sin^5 t; \quad 0 \le t \le 2\pi$

41. $x = |3 \sin t|, \quad y = |3 \cos t|; \quad 0 \le t \le \pi$

42. $x = 3 \sin 2t, \quad y = 3 \cos t; \quad 0 \le t \le 2\pi$

Exercises 43–54: Convert the given equation to parametric equations. Answers may vary.

43. $2x + y = 4$ **44.** $5x - 4y = 20$

45. $y = 4 - x^2$ **46.** $x = y^2 - 2$

47. $x = y^2 + y - 3$ **48.** $5x = y^3 + 1$

49. $x^2 + y^2 = 4$ **50.** $x^2 + y^2 = 9$

51. $\ln y = 0.1x^2$ **52.** $e^x = |1 - y|$

53. $x = y^2 - 2y + 1$ **54.** $x = 4y^2 + 4y + 1$

Exercises 55–60: Graph each pair of parametric equations for $0 \le t \le 2\pi$. Describe any differences in the two graphs.

55. **(a)** $x = 3\cos t, \quad y = 3\sin t$
(b) $x = 3\cos 2t, \quad y = 3\sin 2t$

56. **(a)** $x = 2\cos t, \quad y = 2\sin t$
(b) $x = 2\cos t, \quad y = -2\sin t$

57. **(a)** $x = 3\cos t, \quad y = 3\sin t$
(b) $x = 3\sin t, \quad y = 3\cos t$

58. **(a)** $x = t, \quad y = t^2$
(b) $x = t^2, \quad y = t$

59. **(a)** $x = -1 + \cos t, \quad y = 2 + \sin t$
(b) $x = 1 + \cos t, \quad y = 2 + \sin t$

60. **(a)** $x = 2\cos \frac{1}{2}t, \quad y = 2\sin \frac{1}{2}t$
(b) $x = 2\cos t, \quad y = 2\sin t$

Designing Shapes and Figures

Exercises 61–64: Graph the following set of parametric equations for $0 \le t \le 2\pi$ in the viewing rectangle $[0, 6, 1]$ by $[0, 4, 1]$. Identify the letter of the alphabet that is graphed.

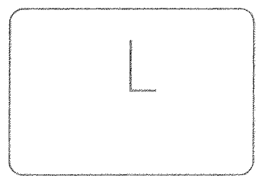
61. $x_1 = 1, \quad y_1 = 1 + t/\pi$
$x_2 = 1 + t/(3\pi), \quad y_2 = 2$
$x_3 = 1 + t/(2\pi), \quad y_3 = 3$

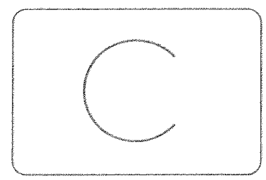
62. $x_1 = 1, \quad y_1 = 1 + t/\pi$
$x_2 = 1 + t/(3\pi), \quad y_2 = 2$
$x_3 = 1 + t/(2\pi), \quad y_3 = 3$
$x_4 = 1 + t/(2\pi), \quad y_4 = 1$

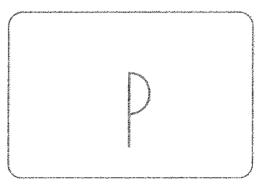
63. $x_1 = 1, \quad y_1 = 1 + t/\pi$
$x_2 = 1 + 1.3\sin(0.5t), \quad y_2 = 2 + \cos(0.5t)$

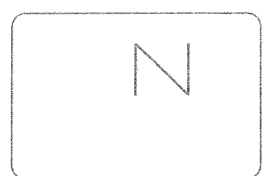
64. $x_1 = 2 + 0.8\cos(0.85t), \quad y_1 = 2 + \sin(0.85t)$
$x_2 = 1.2 + t/(1.3\pi), \quad y_2 = 2$

Exercises 65–68: **Designing Letters** *Find a set of parametric equations that results in a letter similar to the one shown in the figure. Use the viewing rectangle given by $[-4.7, 4.7, 1]$ by $[-3.1, 3.1, 1]$ and turn off the coordinate axes. Answers may vary.*

65.


66.


67.


68.


69. Designing a Face (Refer to Example 9.) Use parametric equations to create your own "smiley" face. This face should have a head, a mouth, and eyes.

70. Designing a Face Add a nose to the face that you designed in Exercise 69.

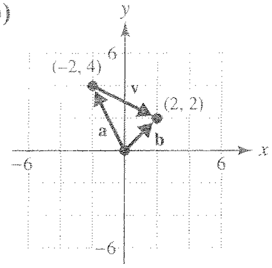
Applications

71. Flight of a Golf Ball (Refer to Example 7.) Two golf balls are hit into the air at 66 feet per second (45 mi/hr), making angles of 35° and 50° with the horizontal. If the ground is level, estimate the horizontal distance traveled by each golf ball.

72. Flight of a Golf Ball Solve Exercise 71 if, instead of the ground being level, the ground is inclined with a slope of $m = 0.1$.

73. Flight of a Golf Ball If a golf ball is hit at 88 feet per second (60 mi/hr), making an angle of 45° with the horizontal, will it go over a fence 10 feet high that is 200 feet away on level ground?

71. **(a)** $\langle -4, 16\rangle$ **(b)** $\langle -12, 0\rangle$ **(c)** $\langle 8, -8\rangle$
73. **(a)** $\langle 8, 0\rangle$ **(b)** $\langle 0, 16\rangle$ **(c)** $\langle -4, -8\rangle$
75. **(a)** $\langle 0, 12\rangle$ **(b)** $\langle -16, -4\rangle$ **(c)** $\langle 8, -4\rangle$
77. **(a)** 2 **(b)** $4\mathbf{i}$ **(c)** $7\mathbf{i} + 3\mathbf{j}$
79. **(a)** $\sqrt{5}$ **(b)** $\langle -2, 4\rangle$ **(c)** $\langle 7, 4\rangle$
81. **(a)** $-21\mathbf{i} + 10\mathbf{j}$ **(b)** $17\mathbf{i} - 10\mathbf{j}$
83. **(a)** $\langle -13, 24\rangle$ **(b)** $\langle 11, -23\rangle$
85. **(a)** $\langle 43, -2\rangle$ **(b)** $\langle -41, -1\rangle$
87. **(a)** 1 **(b)** 81.9° **(c)** Neither
89. **(a)** 0 **(b)** 90° **(c)** Perpendicular
91. **(a)** 122 **(b)** 0° **(c)** Parallel, same direction
93. **(a)** -4 **(b)** 143.1° **(c)** Neither
95. 150 ft-lb **97.** 100,000 ft-lb
99. Work = 590 ft-lb, $\|\mathbf{F}\| = \sqrt{500} \approx 22.4$ lb
101. Work = 27 ft-lb, $\|\mathbf{F}\| = \sqrt{34} \approx 5.8$ lb
103. 24 **105.** 4
107. $\mathbf{v} = \langle 2, 3\rangle$, speed $= \sqrt{13} \approx 3.6$ mi/hr
109. $\mathbf{v} \approx \langle -364.6, -35.4\rangle$, groundspeed ≈ 366.3 mi/hr, bearing $\approx 264.5°$
111. Ground speed ≈ 431.3 mi/hr, bearing $\approx 159.1°$
113. Airspeed ≈ 149.3 mi/hr, groundspeed ≈ 154.6 mi/hr
115. **(a)** $\|\mathbf{R}\| = \sqrt{5} \approx 2.2$, $\|\mathbf{A}\| = \sqrt{1.25} \approx 1.1$. About 2.2 inches of rain fell. The area of the opening of the rain gauge is about 1.1 square inches.
(b) $V = 1.5$; the volume of rain collected in the gauge was 1.5 cubic inches. **(c)** **R** and **A** must be parallel and point in opposite directions.
117. **(a)** $\mathbf{c} = \mathbf{a} + \mathbf{b} = \langle 1, 4\rangle$ **(b)** $\sqrt{17} \approx 4.1$ ft
(c) $3\mathbf{a} + \frac{1}{2}\mathbf{b} = \langle 8, 7\rangle$
119. **(a)** (2, 2) **(b)**



121. $W \approx 297{,}228$ ft-lb
123. Speed ≈ 180 mi/hr, bearing $\approx 128.2°$

8.3 EXTENDED AND DISCOVERY EXERCISES (pp. 659–660)

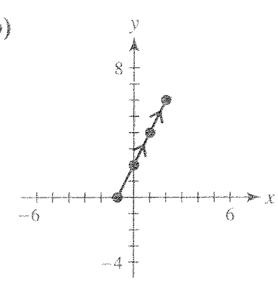
1. Blue **3.** **(a)** Blue **(b)** White

SECTION 8.4 (pp. 667–669)

1. **(a)**

t	0	1	2	3
x	-1	0	1	2
y	0	2	4	6

(b)

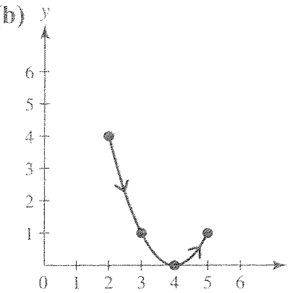


(c) Line segment

3. **(a)**

t	0	1	2	3
x	2	3	4	5
y	4	1	0	1

(b)

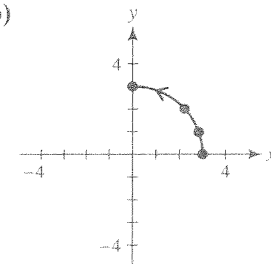


(c) Lower portion of a parabola

5. **(a)**

t	0	1	2	3
x	3	$\sqrt{8}$	$\sqrt{5}$	0
y	0	1	2	3

(b)

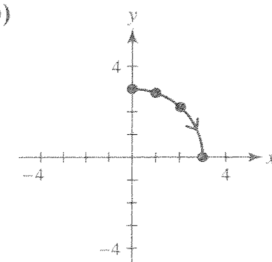


(c) Portion of a circle with radius 3

7. **(a)**

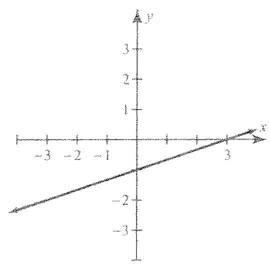
t	0	1	2	3
x	0	1	2	3
y	3	$\sqrt{8}$	$\sqrt{5}$	0

(b)

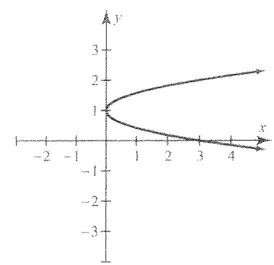


(c) Portion of a circle with radius 3

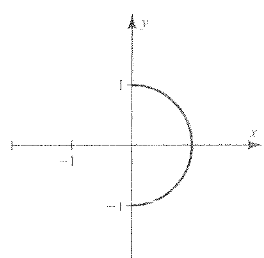
9. $y = \frac{1}{3}x - 1$; line



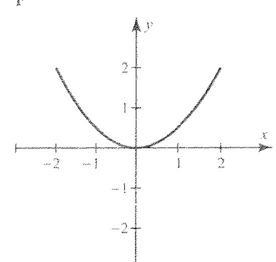
11. $x = 3(y - 1)^2$; parabola



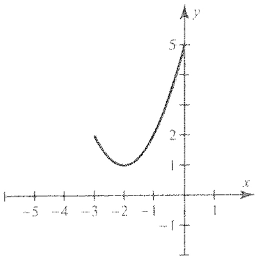
13. $x = \sqrt{1 - y^2}$; portion of a circle with radius 1



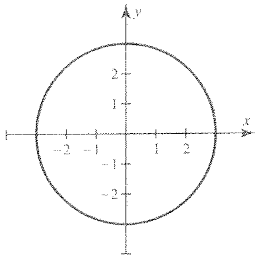
15. $y = \frac{1}{2}x^2$; portion of a parabola



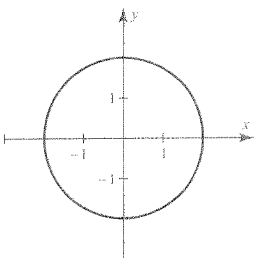
17. $y = x^2 + 4x + 5$; portion of a parabola



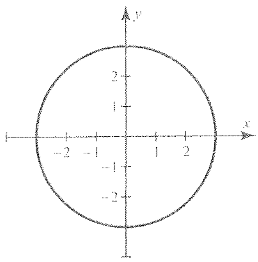
19. $x^2 + y^2 = 9$; circle with radius 3



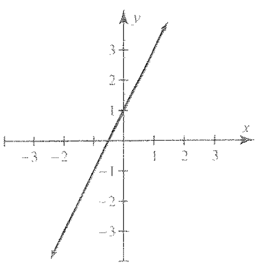
21. $x^2 + y^2 = 4$; circle with radius 2



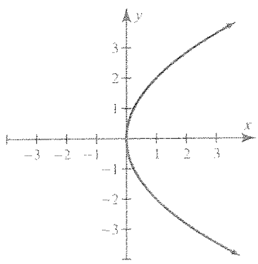
23. $x^2 + y^2 = 9$; circle with radius 3



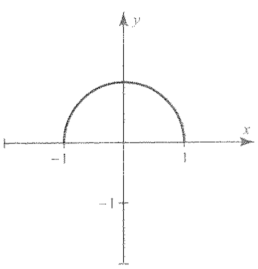
25.



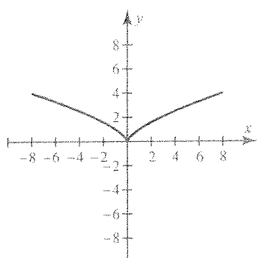
27.



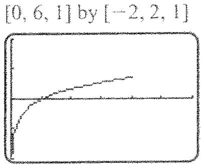
29.



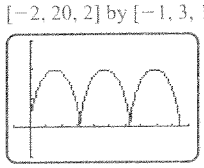
31.



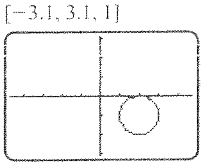
33. [0, 6, 1] by [−2, 2, 1]



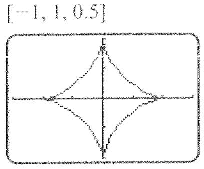
35. [−2, 20, 2] by [−1, 3, 1]



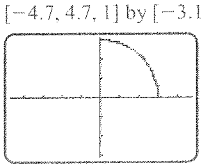
37. [−4.7, 4.7, 1] by [−3.1, 3.1, 1]



39. [−1.5, 1.5, 0.5] by [−1, 1, 0.5]

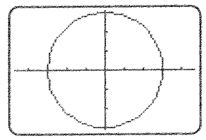


41. [−4.7, 4.7, 1] by [−3.1, 3.1, 1]

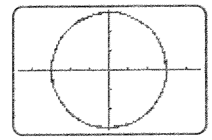


43. $x = t, y = 4 - 2t$ **45.** $x = t, y = 4 - t^2$
47. $x = t^2 + t - 3, y = t$ **49.** $x = 2\cos t, y = 2\sin t$
51. $x = t, y = e^{0.1t^2}$ **53.** $x = t^2 - 2t + 1, y = t$

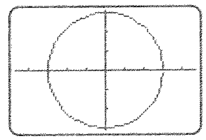
55. **(a)** The curve traces a circle of radius 3 once.
[−4.7, 4.7, 1] by [−3.1, 3.1, 1]



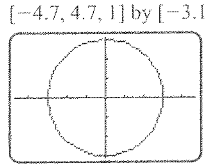
(b) The curve traces a circle of radius 3 twice.
[−4.7, 4.7, 1] by [−3.1, 3.1, 1]



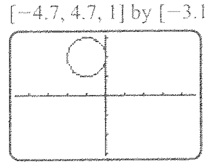
57. **(a)** The curve traces a circle of radius 3 once counter-clockwise, starting at (3, 0).
[−4.7, 4.7, 1] by [−3.1, 3.1, 1]



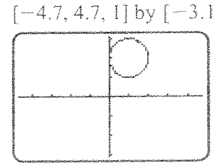
(b) The curve traces a circle of radius 3 once clockwise, starting at (0, 3).
[−4.7, 4.7, 1] by [−3.1, 3.1, 1]



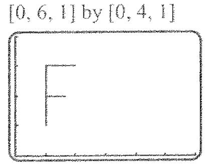
59. **(a)** Circle of radius 1 centered at (−1, 2)
[−4.7, 4.7, 1] by [−3.1, 3.1, 1]



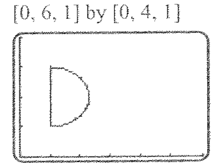
(b) Circle of radius 1 centered at (1, 2)
[−4.7, 4.7, 1] by [−3.1, 3.1, 1]



61. F
[0, 6, 1] by [0, 4, 1]

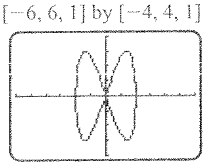


63. D
[0, 6, 1] by [0, 4, 1]

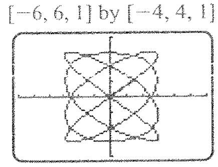


65. $x_1 = 0, y_1 = 2t; x_2 = t, y_2 = 0; 0 \le t \le 1$
67. $x_1 = \sin t,\ y_1 = \cos t;\ x_2 = 0, y_2 = t - 2; 0 \le t \le \pi$
69. Answers may vary.
71. The ball hit at 35° travels about 128 feet. The ball hit at 50° travels about 134 feet.
73. Yes **75.** About 285 feet

77. [−6, 6, 1] by [−4, 4, 1]



79. [−6, 6, 1] by [−4, 4, 1]



8.4 EXTENDED AND DISCOVERY EXERCISES (p. 669)

1. $F_2 = 100$ **3.** $F_2 = 300$