8.3 Polar Form of Complex Numbers; De Moivre's Theorem

Graphing Complex Numbers ▶ Polar Form of Complex Numbers ▶ De Moivre's Theorem ▶ *n*th Roots of Complex Numbers

In this section we represent complex numbers in polar (or trigonometric) form. This enables us to find the *n*th roots of complex numbers. To describe the polar form of complex numbers, we must first learn to work with complex numbers graphically.

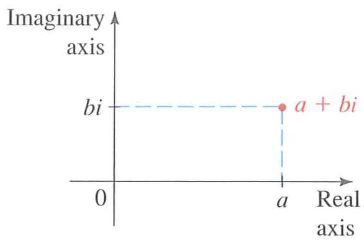


FIGURE 1

▼ Graphing Complex Numbers

To graph real numbers or sets of real numbers, we have been using the number line, which has just one dimension. Complex numbers, however, have two components: a real part and an imaginary part. This suggests that we need two axes to graph complex numbers: one for the real part and one for the imaginary part. We call these the **real axis** and the **imaginary axis**, respectively. The plane determined by these two axes is called the **complex plane**. To graph the complex number $a + bi$, we plot the ordered pair of numbers (a, b) in this plane, as indicated in Figure 1.

EXAMPLE 1 | Graphing Complex Numbers

Graph the complex numbers $z_1 = 2 + 3i$, $z_2 = 3 - 2i$, and $z_1 + z_2$.

SOLUTION We have $z_1 + z_2 = (2 + 3i) + (3 - 2i) = 5 + i$. The graph is shown in Figure 2.

NOW TRY EXERCISE **19** ■

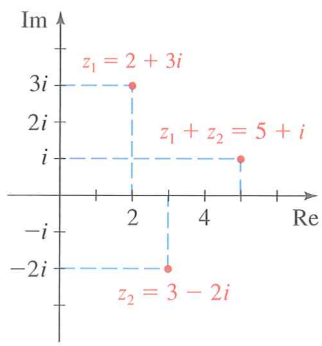


FIGURE 2

EXAMPLE 2 | Graphing Sets of Complex Numbers

Graph each set of complex numbers.

(a) $S = \{a + bi \mid a \geq 0\}$

(b) $T = \{a + bi \mid a < 1, b \geq 0\}$

SOLUTION

(a) S is the set of complex numbers whose real part is nonnegative. The graph is shown in Figure 3(a).

(b) T is the set of complex numbers for which the real part is less than 1 and the imaginary part is nonnegative. The graph is shown in Figure 3(b).

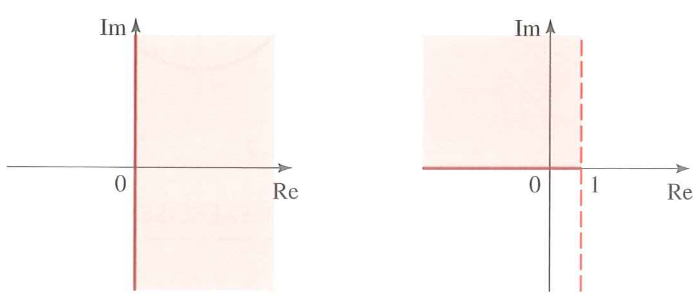


FIGURE 3 (a) (b)

NOW TRY EXERCISE **21** ■

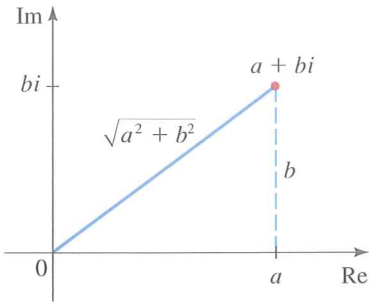


FIGURE 4

Recall that the absolute value of a real number can be thought of as its distance from the origin on the real number line (see Section 1.1). We define absolute value for complex numbers in a similar fashion. Using the Pythagorean Theorem, we can see from Figure 4 that the distance between $a + bi$ and the origin in the complex plane is $\sqrt{a^2 + b^2}$. This leads to the following definition.

MODULUS OF A COMPLEX NUMBER

The **modulus** (or **absolute value**) of the complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}$$

EXAMPLE 3 | Calculating the Modulus

The plural of *modulus* is *moduli*.

Find the moduli of the complex numbers $3 + 4i$ and $8 - 5i$.

SOLUTION

$$|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$
$$|8 - 5i| = \sqrt{8^2 + (-5)^2} = \sqrt{89}$$

NOW TRY EXERCISE **9** ■

EXAMPLE 4 | Absolute Value of Complex Numbers

Graph each set of complex numbers.

(a) $C = \{z \,|\, |z| = 1\}$ **(b)** $D = \{z \,|\, |z| \le 1\}$

SOLUTION

(a) C is the set of complex numbers whose distance from the origin is 1. Thus, C is a circle of radius 1 with center at the origin, as shown in Figure 5.

(b) D is the set of complex numbers whose distance from the origin is less than or equal to 1. Thus, D is the disk that consists of all complex numbers on and inside the circle C of part (a), as shown in Figure 6.

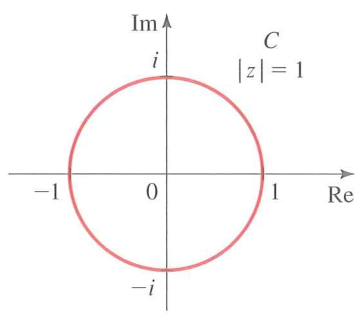


FIGURE 5

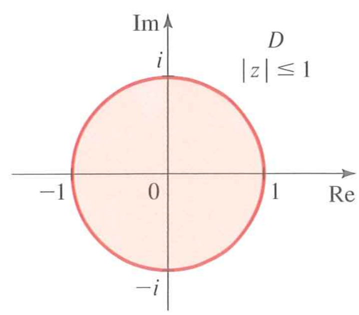


FIGURE 6

NOW TRY EXERCISES **23** AND **25** ■

▼ Polar Form of Complex Numbers

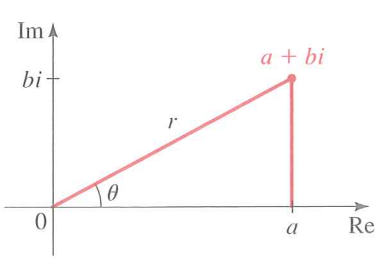


FIGURE 7

Let $z = a + bi$ be a complex number, and in the complex plane let's draw the line segment joining the origin to the point $a + bi$ (see Figure 7). The length of this line segment is $r = |z| = \sqrt{a^2 + b^2}$. If θ is an angle in standard position whose terminal side coincides with this line segment, then by the definitions of sine and cosine (see Section 6.2)

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

so $z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$. We have shown the following.

POLAR FORM OF COMPLEX NUMBERS

A complex number $z = a + bi$ has the **polar form** (or **trigonometric form**)

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is an **argument** of z.

The argument of z is not unique, but any two arguments of z differ by a multiple of 2π. When determining the argument, we must consider the quadrant in which z lies, as we see in the next example.

EXAMPLE 5 | Writing Complex Numbers in Polar Form

Write each complex number in polar form.

(a) $1 + i$ **(b)** $-1 + \sqrt{3}i$ **(c)** $-4\sqrt{3} - 4i$ **(d)** $3 + 4i$

SOLUTION These complex numbers are graphed in Figure 8, which helps us find their arguments.

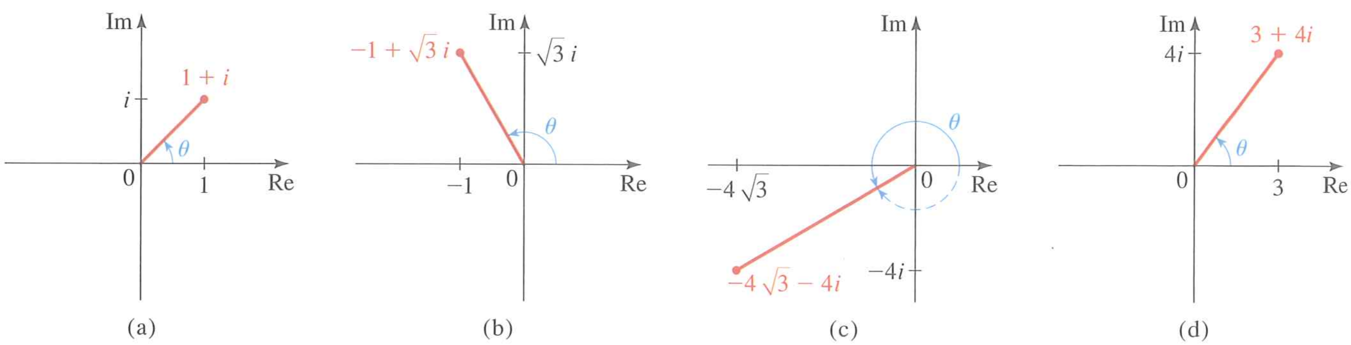


FIGURE 8

$\tan\theta = \frac{1}{1} = 1$
$\theta = \frac{\pi}{4}$

(a) An argument is $\theta = \pi/4$ and $r = \sqrt{1 + 1} = \sqrt{2}$. Thus

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$\tan\theta = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$
$\theta = \frac{2\pi}{3}$

(b) An argument is $\theta = 2\pi/3$ and $r = \sqrt{1 + 3} = 2$. Thus

$$-1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$\tan\theta = \dfrac{-4}{-4\sqrt{3}} = \dfrac{1}{\sqrt{3}}$
$\theta = \frac{7\pi}{6}$

(c) An argument is $\theta = 7\pi/6$ (or we could use $\theta = -5\pi/6$), and $r = \sqrt{48 + 16} = 8$. Thus

$$-4\sqrt{3} - 4i = 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$\tan\theta = \frac{4}{3}$
$\theta = \tan^{-1}\frac{4}{3}$

(d) An argument is $\theta = \tan^{-1}\frac{4}{3}$ and $r = \sqrt{3^2 + 4^2} = 5$. So

$$3 + 4i = 5\left[\cos\left(\tan^{-1}\tfrac{4}{3}\right) + i\sin\left(\tan^{-1}\tfrac{4}{3}\right)\right]$$

. NOW TRY EXERCISES **29**, **31**, AND **33** ■

The Addition Formulas for Sine and Cosine that we discussed in Section 7.2 greatly simplify the multiplication and division of complex numbers in polar form. The following theorem shows how.

MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS

If the two complex numbers z_1 and z_2 have the polar forms

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{Multiplication}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad (z_2 \neq 0) \qquad \text{Division}$$

This theorem says:

To multiply two complex numbers, multiply the moduli and add the arguments.

To divide two complex numbers, divide the moduli and subtract the arguments.

PROOF To prove the Multiplication Formula, we simply multiply the two complex numbers:

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

In the last step we used the Addition Formulas for Sine and Cosine.

The proof of the Division Formula is left as an exercise. ■

EXAMPLE 6 | Multiplying and Dividing Complex Numbers

Let

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \qquad \text{and} \qquad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Find **(a)** $z_1 z_2$ and **(b)** z_1/z_2.

SOLUTION

(a) By the Multiplication Formula

$$\begin{aligned} z_1 z_2 &= (2)(5)\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\right] \\ &= 10\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) \end{aligned}$$

To approximate the answer, we use a calculator in radian mode and get

$$\begin{aligned} z_1 z_2 &\approx 10(-0.2588 + 0.9659i) \\ &= -2.588 + 9.659i \end{aligned}$$

(b) By the Division Formula

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{2}{5}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right]\\ &= \frac{2}{5}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right]\\ &= \frac{2}{5}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)\end{aligned}$$

Using a calculator in radian mode, we get the approximate answer:

$$\frac{z_1}{z_2} \approx \tfrac{2}{5}(0.9659 - 0.2588i) = 0.3864 - 0.1035i$$

NOW TRY EXERCISE 55

De Moivre's Theorem

Repeated use of the Multiplication Formula gives the following useful formula for raising a complex number to a power n for any positive integer n.

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta)$, then for any integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

This theorem says: *To take the nth power of a complex number, we take the nth power of the modulus and multiply the argument by n.*

PROOF By the Multiplication Formula

$$\begin{aligned}z^2 = zz &= r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)]\\ &= r^2(\cos 2\theta + i\sin 2\theta)\end{aligned}$$

Now we multiply z^2 by z to get

$$\begin{aligned}z^3 = z^2 z &= r^3[\cos(2\theta + \theta) + i\sin(2\theta + \theta)]\\ &= r^3(\cos 3\theta + i\sin 3\theta)\end{aligned}$$

Repeating this argument, we see that for any positive integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

A similar argument using the Division Formula shows that this also holds for negative integers.

EXAMPLE 7 | Finding a Power Using De Moivre's Theorem

Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

SOLUTION Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 5(a) that

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

So by De Moivre's Theorem

$$\left(\frac{1}{2}+\frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$

$$= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i$$

NOW TRY EXERCISE 69 ■

▼ *n*th Roots of Complex Numbers

An ***n*th root** of a complex number z is any complex number w such that $w^n = z$. De Moivre's Theorem gives us a method for calculating the nth roots of any complex number.

*n*th ROOTS OF COMPLEX NUMBERS

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n-1$.

PROOF To find the nth roots of z, we need to find a complex number w such that

$$w^n = z$$

Let's write z in polar form:

$$z = r(\cos\theta + i\sin\theta)$$

One nth root of z is

$$w = r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

since by De Moivre's Theorem, $w^n = z$. But the argument θ of z can be replaced by $\theta + 2k\pi$ for any integer k. Since this expression gives a different value of w for $k = 0, 1, 2, \ldots, n-1$, we have proved the formula in the theorem. ■

The following observations help us use the preceding formula.

FINDING THE *n*th ROOTS OF $z = r(\cos\theta + i\sin\theta)$

1. The modulus of each nth root is $r^{1/n}$.
2. The argument of the first root is θ/n.
3. We repeatedly add $2\pi/n$ to get the argument of each successive root.

These observations show that, when graphed, the nth roots of z are spaced equally on the circle of radius $r^{1/n}$.

EXAMPLE 8 | Finding Roots of a Complex Number

Find the six sixth roots of $z = -64$, and graph these roots in the complex plane.

SOLUTION In polar form, $z = 64(\cos\pi + i\sin\pi)$. Applying the formula for nth roots with $n = 6$, we get

$$w_k = 64^{1/6}\left[\cos\left(\frac{\pi + 2k\pi}{6}\right) + i\sin\left(\frac{\pi + 2k\pi}{6}\right)\right]$$

for $k = 0, 1, 2, 3, 4, 5$. Using $64^{1/6} = 2$, we find that the six sixth roots of -64 are

We add $2\pi/6 = \pi/3$ to each argument to get the argument of the next root.

$$w_0 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$$

$$w_1 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i$$

$$w_2 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$$

$$w_3 = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\sqrt{3} - i$$

$$w_4 = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2i$$

$$w_5 = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{3} - i$$

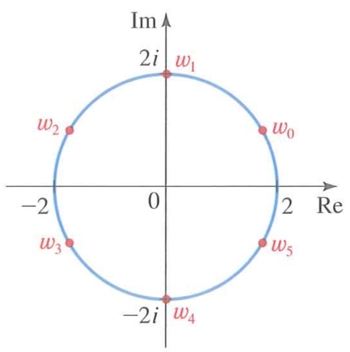


FIGURE 9 The six sixth roots of $z = -64$

All these points lie on a circle of radius 2, as shown in Figure 9.

NOW TRY EXERCISE **85**

When finding roots of complex numbers, we sometimes write the argument θ of the complex number in degrees. In this case the nth roots are obtained from the formula

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n - 1$.

EXAMPLE 9 | Finding Cube Roots of a Complex Number

Find the three cube roots of $z = 2 + 2i$, and graph these roots in the complex plane.

SOLUTION First we write z in polar form using degrees. We have $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ and $\theta = 45°$. Thus

$$z = 2\sqrt{2}(\cos 45° + i\sin 45°)$$

We add $360°/3 = 120°$ to each argument to get the argument of the next root.

Applying the formula for nth roots (in degrees) with $n = 3$, we find that the cube roots of z are of the form

$$w_k = (2\sqrt{2})^{1/3}\left[\cos\left(\frac{45° + 360°k}{3}\right) + i\sin\left(\frac{45° + 360°k}{3}\right)\right]$$

where $k = 0, 1, 2$. Thus the three cube roots are

$$w_0 = \sqrt{2}(\cos 15° + i\sin 15°) \approx 1.366 + 0.366i$$

$(2\sqrt{2})^{1/3} = (2^{3/2})^{1/3} = 2^{1/2} = \sqrt{2}$

$$w_1 = \sqrt{2}(\cos 135° + i\sin 135°) = -1 + i$$

$$w_2 = \sqrt{2}(\cos 255° + i\sin 255°) \approx -0.366 - 1.366i$$

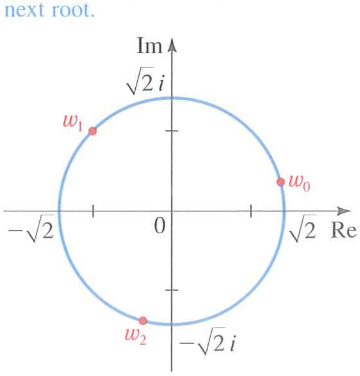


FIGURE 10 The three cube roots of $z = 2 + 2i$

The three cube roots of z are graphed in Figure 10. These roots are spaced equally on a circle of radius $\sqrt{2}$.

NOW TRY EXERCISE **81**

EXAMPLE 10 | Solving an Equation Using the nth Roots Formula

Solve the equation $z^6 + 64 = 0$.

SOLUTION This equation can be written as $z^6 = -64$. Thus the solutions are the sixth roots of -64, which we found in Example 8.

NOW TRY EXERCISE **91**

8.3 EXERCISES

CONCEPTS

1. A complex number $z = a + bi$ has two parts: a is the ________ part, and b is the ________ part. To graph $a + bi$, we graph the ordered pair (,) in the complex plane.

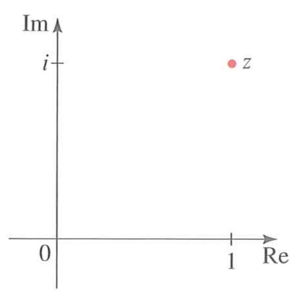
2. Let $z = a + bi$.

(a) The modulus of z is $r =$ ________, and an argument of z is an angle θ satisfying $\tan\theta =$ ________.

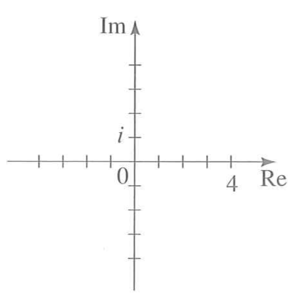
(b) We can express z in polar form as $z =$ ____________, where r is the modulus of z and θ is the argument of z.

3. (a) The complex number $z = -1 + i$ in polar form is $z =$ ________. The complex number $z = 2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$ in rectangular form is $z =$ ________.

(b) The complex number graphed below can be expressed in rectangular form as ______ or in polar form as ______.



4. How many different nth roots does a nonzero complex number have? ________. The number 16 has ________ fourth roots. These roots are ________, ________, ________, and ________. In the complex plane these roots all lie on a circle of radius ________. Graph the roots on the following graph.



SKILLS

5–14 ■ Graph the complex number and find its modulus.

5. $4i$ **6.** $-3i$

7. -2 **8.** 6

9. $5 + 2i$ **10.** $7 - 3i$

11. $\sqrt{3} + i$ **12.** $-1 - \dfrac{\sqrt{3}}{3}i$

13. $\dfrac{3 + 4i}{5}$ **14.** $\dfrac{-\sqrt{2} + i\sqrt{2}}{2}$

15–16 ■ Sketch the complex number z, and also sketch $2z$, $-z$, and $\frac{1}{2}z$ on the same complex plane.

15. $z = 1 + i$ **16.** $z = -1 + i\sqrt{3}$

17–18 ■ Sketch the complex number z and its complex conjugate $\overline{z}$ on the same complex plane.

17. $z = 8 + 2i$ **18.** $z = -5 + 6i$

19–20 ■ Sketch z_1, z_2, $z_1 + z_2$, and z_1z_2 on the same complex plane.

19. $z_1 = 2 - i,\quad z_2 = 2 + i$

20. $z_1 = -1 + i,\quad z_2 = 2 - 3i$

21–28 ■ Sketch the set in the complex plane.

21. $\{z = a + bi \mid a \le 0, b \ge 0\}$

22. $\{z = a + bi \mid a > 1, b > 1\}$

23. $\{z \mid |z| = 3\}$ **24.** $\{z \mid |z| \ge 1\}$

25. $\{z \mid |z| < 2\}$ **26.** $\{z \mid 2 \le |z| \le 5\}$

27. $\{z = a + bi \mid a + b < 2\}$

28. $\{z = a + bi \mid a \ge b\}$

29–52 ■ Write the complex number in polar form with argument θ between 0 and 2π.

29. $1 + i$ **30.** $1 + \sqrt{3}i$ **31.** $\sqrt{2} - \sqrt{2}i$

32. $1 - i$ **33.** $2\sqrt{3} - 2i$ **34.** $-1 + i$

35. $-3i$ **36.** $-3 - 3\sqrt{3}i$ **37.** $5 + 5i$

38. 4 **39.** $4\sqrt{3} - 4i$ **40.** $8i$

41. -20 **42.** $\sqrt{3} + i$ **43.** $3 + 4i$

44. $i(2 - 2i)$ **45.** $3i(1 + i)$ **46.** $2(1 - i)$

47. $4(\sqrt{3}+i)$ **48.** $-3-3i$ **49.** $2+i$

50. $3+\sqrt{3}\,i$ **51.** $\sqrt{2}+\sqrt{2}\,i$ **52.** $-\pi i$

53–60 ■ Find the product z_1z_2 and the quotient z_1/z_2. Express your answer in polar form.

53. $z_1 = \cos\pi + i\sin\pi,\quad z_2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

54. $z_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4},\quad z_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

55. $z_1 = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right),\quad z_2 = 5\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

56. $z_1 = 7\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right),\quad z_2 = 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$

57. $z_1 = 4(\cos 120° + i\sin 120°)$,
$z_2 = 2(\cos 30° + i\sin 30°)$

58. $z_1 = \sqrt{2}(\cos 75° + i\sin 75°)$,
$z_2 = 3\sqrt{2}(\cos 60° + i\sin 60°)$

59. $z_1 = 4(\cos 200° + i\sin 200°)$,
$z_2 = 25(\cos 150° + i\sin 150°)$

60. $z_1 = \frac{4}{5}(\cos 25° + i\sin 25°)$,
$z_2 = \frac{1}{5}(\cos 155° + i\sin 155°)$

61–68 ■ Write z_1 and z_2 in polar form, and then find the product z_1z_2 and the quotients z_1/z_2 and $1/z_1$.

61. $z_1 = \sqrt{3}+i,\quad z_2 = 1+\sqrt{3}\,i$

62. $z_1 = \sqrt{2}-\sqrt{2}\,i,\quad z_2 = 1-i$

63. $z_1 = 2\sqrt{3}-2i,\quad z_2 = -1+i$

64. $z_1 = -\sqrt{2}\,i,\quad z_2 = -3-3\sqrt{3}\,i$

65. $z_1 = 5+5i,\quad z_2 = 4$ **66.** $z_1 = 4\sqrt{3}-4i,\quad z_2 = 8i$

67. $z_1 = -20,\quad z_2 = \sqrt{3}+i$ **68.** $z_1 = 3+4i,\quad z_2 = 2-2i$

69–80 ■ Find the indicated power using De Moivre's Theorem.

69. $(1+i)^{20}$ **70.** $(1-\sqrt{3}\,i)^5$

71. $(2\sqrt{3}+2i)^5$ **72.** $(1-i)^8$

73. $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^{12}$ **74.** $(\sqrt{3}-i)^{-10}$

75. $(2-2i)^8$ **76.** $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{15}$

77. $(-1-i)^7$ **78.** $(3+\sqrt{3}\,i)^4$

79. $(2\sqrt{3}+2i)^{-5}$ **80.** $(1-i)^{-8}$

81–90 ■ Find the indicated roots, and graph the roots in the complex plane.

81. The square roots of $4\sqrt{3}+4i$

82. The cube roots of $4\sqrt{3}+4i$

83. The fourth roots of $-81i$

84. The fifth roots of 32

85. The eighth roots of 1

86. The cube roots of $1+i$

87. The cube roots of i

88. The fifth roots of i

89. The fourth roots of -1

90. The fifth roots of $-16-16\sqrt{3}i$

91–96 ■ Solve the equation.

91. $z^4+1=0$ **92.** $z^8-i=0$

93. $z^3-4\sqrt{3}-4i=0$ **94.** $z^6-1=0$

95. $z^3+1=-i$ **96.** $z^3-1=0$

97. **(a)** Let $w = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ where n is a positive integer. Show that $1, w, w^2, w^3, \ldots, w^{n-1}$ are the n distinct nth roots of 1.

(b) If $z \neq 0$ is any complex number and $s^n = z$, show that the n distinct nth roots of z are

$$s,\ sw,\ sw^2,\ sw^3,\ \ldots,\ sw^{n-1}$$

DISCOVERY ■ DISCUSSION ■ WRITING

98. **Sums of Roots of Unity** Find the exact values of all three cube roots of 1 (see Exercise 97) and then add them. Do the same for the fourth, fifth, sixth, and eighth roots of 1. What do you think is the sum of the nth roots of 1 for any n?

99. **Products of Roots of Unity** Find the product of the three cube roots of 1 (see Exercise 97). Do the same for the fourth, fifth, sixth, and eighth roots of 1. What do you think is the product of the nth roots of 1 for any n?

100. **Complex Coefficients and the Quadratic Formula** The quadratic formula works whether the coefficients of the equation are real or complex. Solve these equations using the quadratic formula and, if necessary, De Moivre's Theorem.

(a) $z^2+(1+i)z+i=0$

(b) $z^2-iz+1=0$

(c) $z^2-(2-i)z-\frac{1}{4}i=0$

DISCOVERY PROJECT **Fractals**

In this project we use graphs of complex numbers to create fractal images. You can find the project at the book companion website: **www.stewartmath.com**

8.4 PLANE CURVES AND PARAMETRIC EQUATIONS

So far, we have described a curve by giving an equation (in rectangular or polar coordinates) that the coordinates of all the points on the curve must satisfy. But not all curves in the plane can be described in this way. In this section we study parametric equations, which are a general method for describing any curve.

▼ Plane Curves and Parametric Equations

We can think of a curve as the path of a point moving in the plane; the x- and y-coordinates of the point are then functions of time. This idea leads to the following definition.

PLANE CURVES AND PARAMETRIC EQUATIONS

If f and g are functions defined on an interval I, then the set of points $(f(t), g(t))$ is a **plane curve**. The equations

$$x = f(t) \qquad y = g(t)$$

where $t \in I$, are **parametric equations** for the curve, with **parameter** t.

EXAMPLE 1 | Sketching a Plane Curve

Sketch the curve defined by the parametric equations

$$x = t^2 - 3t \qquad y = t - 1$$

SOLUTION For every value of t, we get a point on the curve. For example, if $t = 0$, then $x = 0$ and $y = -1$, so the corresponding point is $(0, -1)$. In Figure 1 we plot the points (x, y) determined by the values of t shown in the following table.

t	x	y
-2	10	-3
-1	4	-2
0	0	-1
1	-2	0
2	-2	1
3	0	2
4	4	3
5	10	4

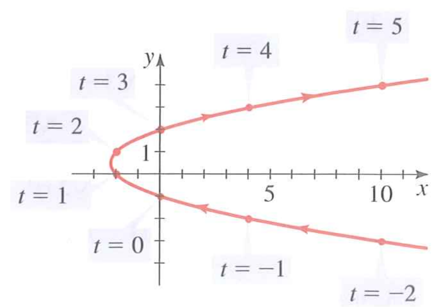


FIGURE 1

As t increases, a particle whose position is given by the parametric equations moves along the curve in the direction of the arrows.

NOW TRY EXERCISE 3 ■