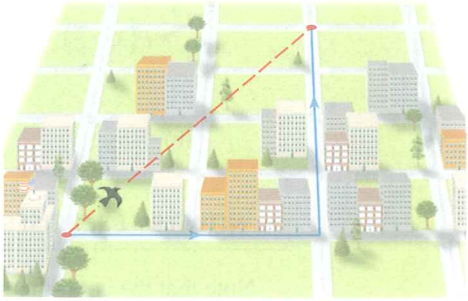
CHAPTER 8

Polar Coordinates and Parametric Equations

In Section 1.8 we learned how to graph points in rectangular coordinates. In this chapter we study a different way of locating points in the plane, called *polar coordinates.* Using rectangular coordinates is like describing a location in a city by saying that it's at the corner of 2nd Street and 4th Avenue; these directions would help a taxi driver find the location. But we may also describe this same location "as the crow flies"; we can say that it's 1.5 miles northeast of City Hall. So instead of specifying the location with respect to a grid of streets and avenues, we specify it by giving its distance and direction from a fixed reference point. That's what we do in the polar coordinate system. In polar coordinates the location of a point is given by an ordered pair of numbers: the distance of the point from the origin (or pole) and the angle from the positive x-axis.



Why do we study different coordinate systems? It's because certain curves are more naturally described in one coordinate system rather than another. For example, in rectangular coordinates lines and parabolas have simple equations, but equations of circles are rather complicated. We'll see that in polar coordinates circles have very simple equations.

8.1 POLAR COORDINATES

Definition of Polar Coordinates ▶ Relationship Between Polar and Rectangular Coordinates ▶ Polar Equations

In this section we define polar coordinates, and we learn how polar coordinates are related to rectangular coordinates.

▼ Definition of Polar Coordinates

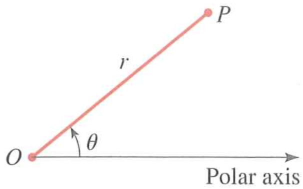


FIGURE 1

The **polar coordinate system** uses distances and directions to specify the location of a point in the plane. To set up this system, we choose a fixed point O in the plane called the **pole** (or **origin**) and draw from O a ray (half-line) called the **polar axis** as in Figure 1. Then each point P can be assigned polar coordinates $P(r, \theta)$ where

r is the *distance* from O to P

θ is the *angle* between the polar axis and the segment $\overline{OP}$

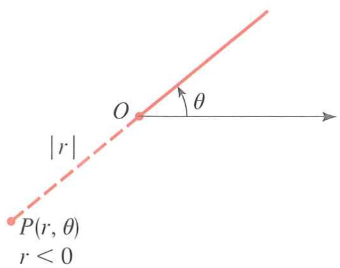


FIGURE 2

We use the convention that θ is positive if measured in a counterclockwise direction from the polar axis or negative if measured in a clockwise direction. If r is negative, then $P(r, \theta)$ is defined to be the point that lies $|r|$ units from the pole in the direction opposite to that given by θ (see Figure 2).

EXAMPLE 1 | Plotting Points in Polar Coordinates

Plot the points whose polar coordinates are given.

(a) $(1, 3\pi/4)$ **(b)** $(3, -\pi/6)$ **(c)** $(3, 3\pi)$ **(d)** $(-4, \pi/4)$

SOLUTION The points are plotted in Figure 3. Note that the point in part (d) lies 4 units from the origin along the angle $5\pi/4$, because the given value of r is negative.

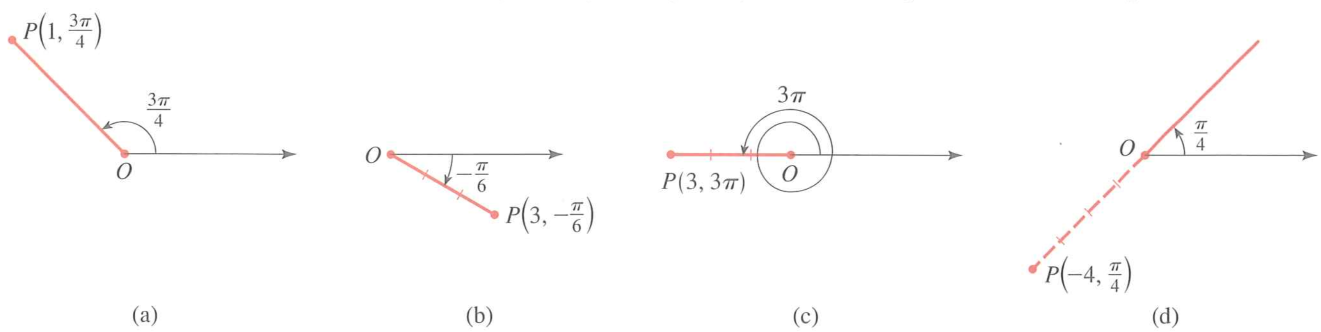


FIGURE 3

NOW TRY EXERCISES **3** AND **5** ■

Note that the coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point, as shown in Figure 4. Moreover, because the angles $\theta + 2n\pi$ (where n is any integer) all have the same terminal side as the angle θ, each point in the plane has infinitely many representations in polar coordinates. In fact, any point $P(r, \theta)$ can also be represented by

$$P(r, \theta + 2n\pi) \qquad \text{and} \qquad P(-r, \theta + (2n + 1)\pi)$$

for any integer n.

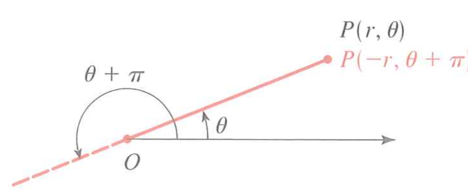


FIGURE 4

EXAMPLE 2 | Different Polar Coordinates for the Same Point

(a) Graph the point with polar coordinates $P(2, \pi/3)$.

(b) Find two other polar coordinate representations of P with $r > 0$ and two with $r < 0$.

SOLUTION

(a) The graph is shown in Figure 5(a).

(b) Other representations with $r > 0$ are

$$\left(2, \frac{\pi}{3} + 2\pi\right) = \left(2, \frac{7\pi}{3}\right) \qquad \text{Add } 2\pi \text{ to } \theta$$

$$\left(2, \frac{\pi}{3} - 2\pi\right) = \left(2, -\frac{5\pi}{3}\right) \qquad \text{Add } -2\pi \text{ to } \theta$$

Other representations with $r < 0$ are

$$\left(-2, \frac{\pi}{3} + \pi\right) = \left(-2, \frac{4\pi}{3}\right) \qquad \text{Replace } r \text{ by } -r \text{ and add } \pi \text{ to } \theta$$

$$\left(-2, \frac{\pi}{3} - \pi\right) = \left(-2, -\frac{2\pi}{3}\right) \qquad \text{Replace } r \text{ by } -r \text{ and add } -\pi \text{ to } \theta$$

The graphs in Figure 5 explain why these coordinates represent the same point.

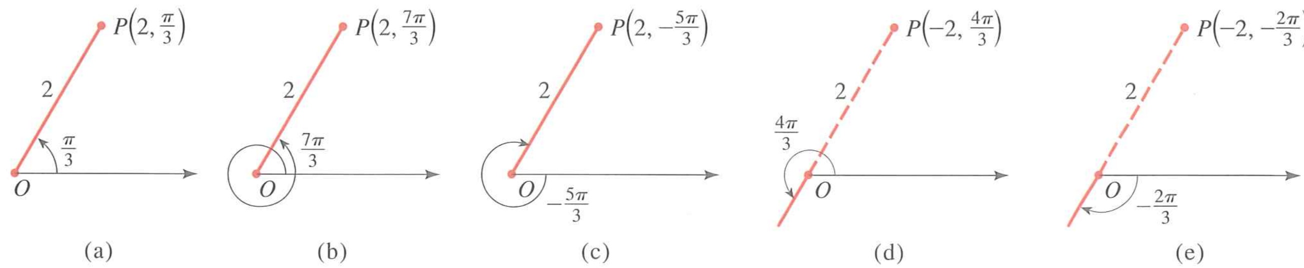


FIGURE 5

NOW TRY EXERCISE 9

Relationship Between Polar and Rectangular Coordinates

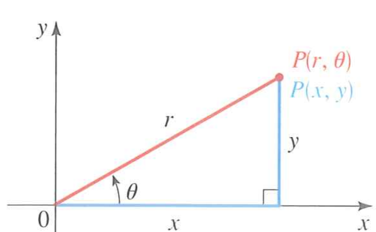


FIGURE 6

Situations often arise in which we need to consider polar and rectangular coordinates simultaneously. The connection between the two systems is illustrated in Figure 6, where the polar axis coincides with the positive x-axis. The formulas in the following box are obtained from the figure using the definitions of the trigonometric functions and the Pythagorean Theorem. (Although we have pictured the case where $r > 0$ and θ is acute, the formulas hold for any angle θ and for any value of r.)

RELATIONSHIP BETWEEN POLAR AND RECTANGULAR COORDINATES

1. To change from polar to rectangular coordinates, use the formulas

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

2. To change from rectangular to polar coordinates, use the formulas

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

EXAMPLE 3 | Converting Polar Coordinates to Rectangular Coordinates

Find rectangular coordinates for the point that has polar coordinates $(4, 2\pi/3)$.

SOLUTION Since $r = 4$ and $\theta = 2\pi/3$, we have

$$x = r\cos\theta = 4\cos\frac{2\pi}{3} = 4\cdot\left(-\frac{1}{2}\right) = -2$$

$$y = r\sin\theta = 4\sin\frac{2\pi}{3} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$$

Thus the point has rectangular coordinates $(-2, 2\sqrt{3})$.

NOW TRY EXERCISE **27**

EXAMPLE 4 | Converting Rectangular Coordinates to Polar Coordinates

Find polar coordinates for the point that has rectangular coordinates $(2, -2)$.

SOLUTION Using $x = 2$, $y = -2$, we get

$$r^2 = x^2 + y^2 = 2^2 + (-2)^2 = 8$$

so $r = 2\sqrt{2}$ or $-2\sqrt{2}$. Also

$$\tan\theta = \frac{y}{x} = \frac{-2}{2} = -1$$

so $\theta = 3\pi/4$ or $-\pi/4$. Since the point $(2, -2)$ lies in Quadrant IV (see Figure 7), we can represent it in polar coordinates as $(2\sqrt{2}, -\pi/4)$ or $(-2\sqrt{2}, 3\pi/4)$.

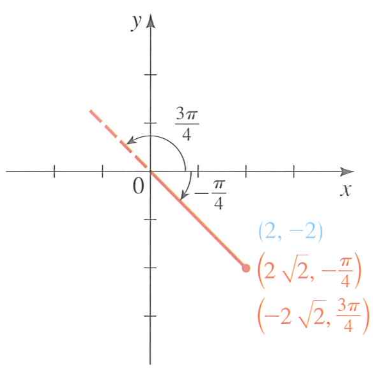


FIGURE 7

NOW TRY EXERCISE **35**

Note that the equations relating polar and rectangular coordinates do not uniquely determine r or θ. When we use these equations to find the polar coordinates of a point, we must be careful that the values we choose for r and θ give us a point in the correct quadrant, as we did in Example 4.

Polar Equations

In Examples 3 and 4 we converted points from one coordinate system to the other. Now we consider the same problem for equations.

EXAMPLE 5 | Converting an Equation from Rectangular to Polar Coordinates

Express the equation $x^2 = 4y$ in polar coordinates.

SOLUTION We use the formulas $x = r\cos\theta$ and $y = r\sin\theta$:

$x^2 = 4y$	Rectangular equation
$(r\cos\theta)^2 = 4(r\sin\theta)$	Substitute $x = r\cos\theta$, $y = r\sin\theta$
$r^2\cos^2\theta = 4r\sin\theta$	Expand
$r = 4\dfrac{\sin\theta}{\cos^2\theta}$	Divide by $r\cos^2\theta$
$r = 4\sec\theta\tan\theta$	Simplify

NOW TRY EXERCISE **45**

As Example 5 shows, converting from rectangular to polar coordinates is straightforward: Just replace x by $r\cos\theta$ and y by $r\sin\theta$, and then simplify. But converting polar equations to rectangular form often requires more thought.

EXAMPLE 6 | Converting Equations from Polar to Rectangular Coordinates

Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

(a) $r = 5\sec\theta$ **(b)** $r = 2\sin\theta$ **(c)** $r = 2 + 2\cos\theta$

SOLUTION

(a) Since $\sec\theta = 1/\cos\theta$, we multiply both sides by $\cos\theta$:

$$r = 5\sec\theta \qquad \text{Polar equation}$$
$$r\cos\theta = 5 \qquad \text{Multiply by } \cos\theta$$
$$x = 5 \qquad \text{Substitute } x = r\cos\theta$$

The graph of $x = 5$ is the vertical line in Figure 8.

(b) We multiply both sides of the equation by r, because then we can use the formulas $r^2 = x^2 + y^2$ and $r\sin\theta = y$:

$$r = 2\sin\theta \qquad \text{Polar equation}$$
$$r^2 = 2r\sin\theta \qquad \text{Multiply by } r$$
$$x^2 + y^2 = 2y \qquad r^2 = x^2 + y^2 \text{ and } r\sin\theta = y$$
$$x^2 + y^2 - 2y = 0 \qquad \text{Subtract } 2y$$
$$x^2 + (y-1)^2 = 1 \qquad \text{Complete the square in } y$$

This is the equation of a circle of radius 1 centered at the point $(0, 1)$. It is graphed in Figure 9.

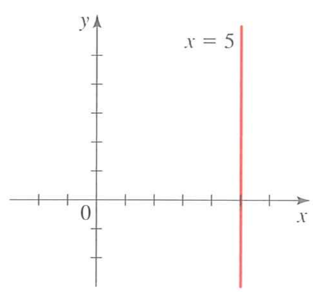


FIGURE 8

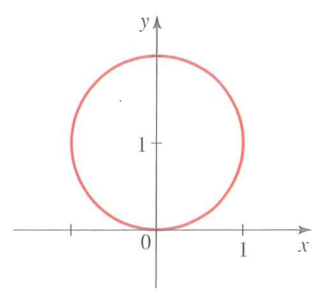


FIGURE 9

(c) We first multiply both sides of the equation by r:

$$r^2 = 2r + 2r\cos\theta$$

Using $r^2 = x^2 + y^2$ and $x = r\cos\theta$, we can convert two terms in the equation into rectangular coordinates, but eliminating the remaining r requires more work:

$$x^2 + y^2 = 2r + 2x \qquad r^2 = x^2 + y^2 \text{ and } r\cos\theta = x$$
$$x^2 + y^2 - 2x = 2r \qquad \text{Subtract } 2x$$
$$(x^2 + y^2 - 2x)^2 = 4r^2 \qquad \text{Square both sides}$$
$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2) \qquad r^2 = x^2 + y^2$$

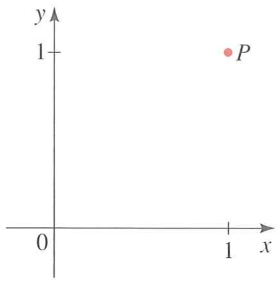
In this case the rectangular equation looks more complicated than the polar equation. Although we cannot easily determine the graph of the equation from its rectangular form, we will see in the next section how to graph it using the polar equation.

NOW TRY EXERCISES **53**, **55**, AND **57**

8.1 EXERCISES

CONCEPTS

1. We can describe the location of a point in the plane using different ________ systems. The point P shown in the figure has rectangular coordinates (,) and polar coordinates (,).



2. Let P be a point in the plane.

(a) If P has polar coordinates (r, θ) then it has rectangular coordinates (x, y) where $x =$ ________ and $y =$ ________.

(b) If P has rectangular coordinates (x, y) then it has polar coordinates (r, θ) where $r^2 =$ ________ and $\tan\theta =$ ________.

SKILLS

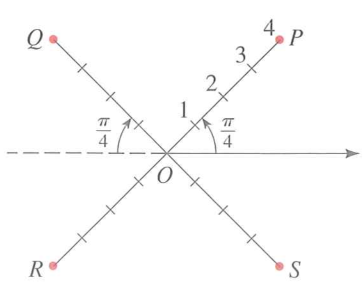
3–8 ■ Plot the point that has the given polar coordinates.

3. $(4, \pi/4)$ **4.** $(1, 0)$ **5.** $(6, -7\pi/6)$
6. $(3, -2\pi/3)$ **7.** $(-2, 4\pi/3)$ **8.** $(-5, -17\pi/6)$

9–14 ■ Plot the point that has the given polar coordinates. Then give two other polar coordinate representations of the point, one with $r < 0$ and the other with $r > 0$.

9. $(3, \pi/2)$ **10.** $(2, 3\pi/4)$ **11.** $(-1, 7\pi/6)$
12. $(-2, -\pi/3)$ **13.** $(-5, 0)$ **14.** $(3, 1)$

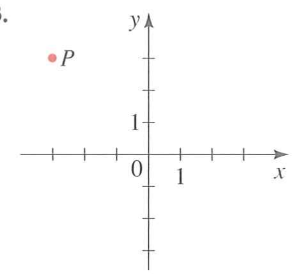
15–22 ■ Determine which point in the figure, P, Q, R, or S, has the given polar coordinates.



15. $(4, 3\pi/4)$ **16.** $(4, -3\pi/4)$
17. $(-4, -\pi/4)$ **18.** $(-4, 13\pi/4)$
19. $(4, -23\pi/4)$ **20.** $(-4, 23\pi/4)$
21. $(-4, 101\pi/4)$ **22.** $(4, 103\pi/4)$

23–24 ■ A point is graphed in rectangular form. Find polar coordinates for the point, with $r > 0$ and $0 < \theta < 2\pi$.

23.



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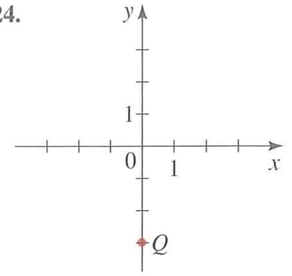
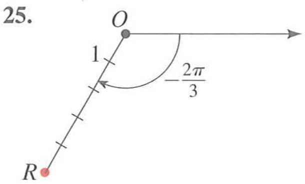


25–26 ■ A point is graphed in polar form. Find its rectangular coordinates.

25.



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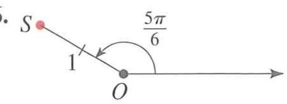

27–34 ■ Find the rectangular coordinates for the point whose polar coordinates are given.

27. $(4, \pi/6)$ **28.** $(6, 2\pi/3)$
29. $(\sqrt{2}, -\pi/4)$ **30.** $(-1, 5\pi/2)$
31. $(5, 5\pi)$ **32.** $(0, 13\pi)$
33. $(6\sqrt{2}, 11\pi/6)$ **34.** $(\sqrt{3}, -5\pi/3)$

35–42 ■ Convert the rectangular coordinates to polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.

35. $(-1, 1)$ **36.** $(3\sqrt{3}, -3)$
37. $(\sqrt{8}, \sqrt{8})$ **38.** $(-\sqrt{6}, -\sqrt{2})$
39. $(3, 4)$ **40.** $(1, -2)$
41. $(-6, 0)$ **42.** $(0, -\sqrt{3})$

43–48 ■ Convert the equation to polar form.

43. $x = y$ **44.** $x^2 + y^2 = 9$
45. $y = x^2$ **46.** $y = 5$
47. $x = 4$ **48.** $x^2 - y^2 = 1$

49–68 ■ Convert the polar equation to rectangular coordinates.

49. $r = 7$ **50.** $r = -3$
51. $\theta = -\dfrac{\pi}{2}$ **52.** $\theta = \pi$
53. $r\cos\theta = 6$ **54.** $r = 2\csc\theta$

55. $r = 4 \sin \theta$

56. $r = 6 \cos \theta$

57. $r = 1 + \cos \theta$

58. $r = 3(1 - \sin \theta)$

59. $r = 1 + 2 \sin \theta$

60. $r = 2 - \cos \theta$

61. $r = \dfrac{1}{\sin \theta - \cos \theta}$

62. $r = \dfrac{1}{1 + \sin \theta}$

63. $r = \dfrac{4}{1 + 2 \sin \theta}$

64. $r = \dfrac{2}{1 - \cos \theta}$

65. $r^2 = \tan \theta$

66. $r^2 = \sin 2\theta$

67. $\sec \theta = 2$

68. $\cos 2\theta = 1$

DISCOVERY ■ DISCUSSION ■ WRITING

69. The Distance Formula in Polar Coordinates

(a) Use the Law of Cosines to prove that the distance between the polar points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$$

(b) Find the distance between the points whose polar coordinates are $(3, 3\pi/4)$ and $(1, 7\pi/6)$, using the formula from part (a).

(c) Now convert the points in part (b) to rectangular coordinates. Find the distance between them using the usual Distance Formula. Do you get the same answer?

8.2 Graphs of Polar Equations

Graphing Polar Equations ▶ Symmetry ▶ Graphing Polar Equations with Graphing Devices

The **graph of a polar equation** $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation. Many curves that arise in mathematics and its applications are more easily and naturally represented by polar equations than by rectangular equations.

▼ Graphing Polar Equations

A rectangular grid is helpful for plotting points in rectangular coordinates (see Figure 1(a)). To plot points in polar coordinates, it is convenient to use a grid consisting of circles centered at the pole and rays emanating from the pole, as in Figure 1(b). We will use such grids to help us sketch polar graphs.

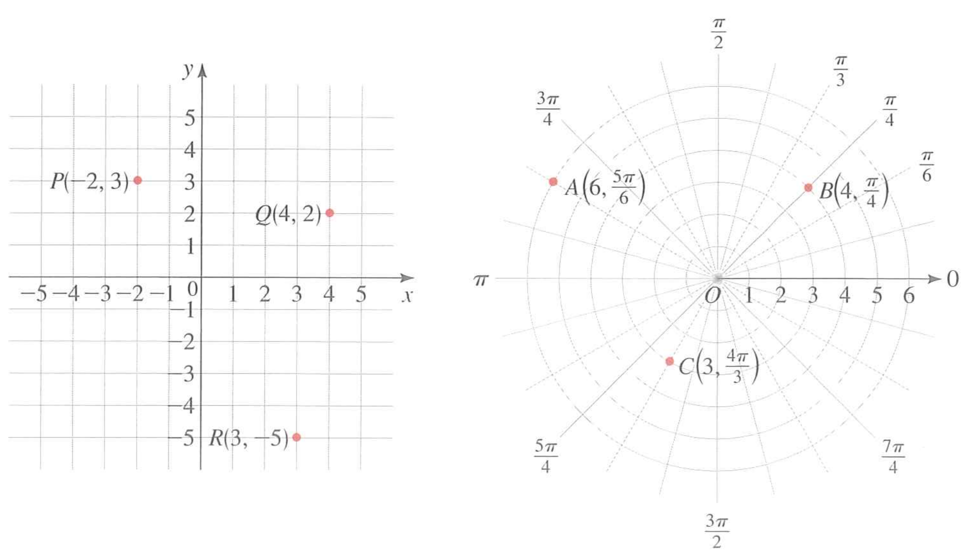

FIGURE 1 (a) Grid for rectangular coordinates (b) Grid for polar coordinates

In Examples 1 and 2 we see that circles centered at the origin and lines that pass through the origin have particularly simple equations in polar coordinates.

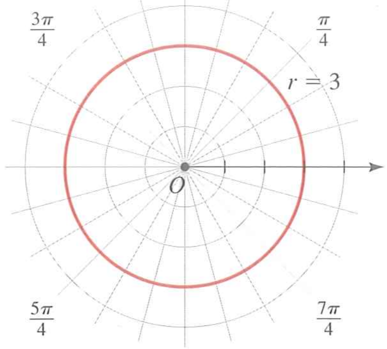

FIGURE 2

EXAMPLE 1 | Sketching the Graph of a Polar Equation

Sketch a graph of the equation $r = 3$, and express the equation in rectangular coordinates.

SOLUTION The graph consists of all points whose r-coordinate is 3, that is, all points that are 3 units away from the origin. So the graph is a circle of radius 3 centered at the origin, as shown in Figure 2.

Squaring both sides of the equation, we get

$$r^2 = 3^2 \qquad \text{Square both sides}$$

$$x^2 + y^2 = 9 \qquad \text{Substitute } r^2 = x^2 + y^2$$

So the equivalent equation in rectangular coordinates is $x^2 + y^2 = 9$.

NOW TRY EXERCISE **17**

In general, the graph of the equation $r = a$ is a circle of radius $|a|$ centered at the origin. Squaring both sides of this equation, we see that the equivalent equation in rectangular coordinates is $x^2 + y^2 = a^2$.

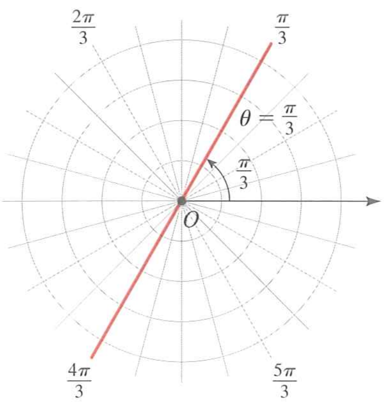

FIGURE 3

EXAMPLE 2 | Sketching the Graph of a Polar Equation

Sketch a graph of the equation $\theta = \pi/3$, and express the equation in rectangular coordinates.

SOLUTION The graph consists of all points whose θ-coordinate is $\pi/3$. This is the straight line that passes through the origin and makes an angle of $\pi/3$ with the polar axis (see Figure 3). Note that the points $(r, \pi/3)$ on the line with $r > 0$ lie in Quadrant I, whereas those with $r < 0$ lie in Quadrant III. If the point (x, y) lies on this line, then

$$\frac{y}{x} = \tan\theta = \tan\frac{\pi}{3} = \sqrt{3}$$

Thus, the rectangular equation of this line is $y = \sqrt{3}x$.

NOW TRY EXERCISE **19**

To sketch a polar curve whose graph isn't as obvious as the ones in the preceding examples, we plot points calculated for sufficiently many values of θ and then join them in a continuous curve. (This is what we did when we first learned to graph functions in rectangular coordinates.)

EXAMPLE 3 | Sketching the Graph of a Polar Equation

Sketch a graph of the polar equation $r = 2\sin\theta$.

SOLUTION We first use the equation to determine the polar coordinates of several points on the curve. The results are shown in the following table.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$r = 2\sin\theta$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

We plot these points in Figure 4 and then join them to sketch the curve. The graph appears to be a circle. We have used values of θ only between 0 and π, since the same points (this time expressed with negative r-coordinates) would be obtained if we allowed θ to range from π to 2π.