7.5 More Trigonometric Equations

Solving Trigonometric Equations by Using Identities > Equations with Trigonometric Functions of Multiples of Angles

In this section we solve trigonometric equations by first using identities to simplify the equation. We also solve trigonometric equations in which the terms contain multiples of angles.

lacktriangle Solving Trigonometric Equations by Using Identities

In the next two examples we use trigonometric identities to express a trigonometric equation in a form in which it can be factored.

EXAMPLE 1 Using a Trigonometric Identity

Solve the equation $1 + \sin \theta = 2\cos^2 \theta$.

SOLUTION We first need to rewrite this equation so that it contains only one trigonometric function. To do this, we use a trigonometric identity:

$$1 + \sin \theta = 2 \cos^2 \theta \qquad \qquad \text{Given equation}$$

$$1 + \sin \theta = 2(1 - \sin^2 \theta) \qquad \qquad \text{Pythagorean identity}$$

$$2 \sin^2 \theta + \sin \theta - 1 = 0 \qquad \qquad \text{Put all terms on one side}$$

$$(2 \sin \theta - 1)(\sin \theta + 1) = 0 \qquad \qquad \text{Factor}$$

$$2 \sin \theta - 1 = 0 \qquad \text{or} \qquad \sin \theta + 1 = 0 \qquad \qquad \text{Set each factor equal to 0}$$

$$\sin \theta = \frac{1}{2} \qquad \text{or} \qquad \sin \theta = -1 \qquad \qquad \text{Solve for } \sin \theta$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \qquad \text{or} \qquad \theta = \frac{3\pi}{2} \qquad \qquad \text{Solve for } \theta \text{ in the interval } [0, 2\pi)$$

Because sine has period 2π , we get all the solutions of the equation by adding integer multiples of 2π to these solutions. Thus the solutions are

interval $[0, 2\pi)$

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi \qquad \theta = \frac{3\pi}{2} + 2k\pi$$

where k is any integer.

NOW TRY EXERCISE 3

EXAMPLE 2 Using a Trigonometric Identity

Solve the equation $\sin 2\theta - \cos \theta = 0$.

50LUTION The first term is a function of 2θ , and the second is a function of θ , so we begin by using a trigonometric identity to rewrite the first term as a function of θ only:

$$\sin 2\theta - \cos \theta = 0$$
 Given equation
 $2 \sin \theta \cos \theta - \cos \theta = 0$ Double-Angle Formula
 $\cos \theta (2 \sin \theta - 1) = 0$ Factor

1

$$\cos\theta=0$$
 or $2\sin\theta-1=0$ Set each factor equal to 0
$$\sin\theta=\frac{1}{2}$$
 Solve for $\sin\theta$
$$\theta=\frac{\pi}{2},\frac{3\pi}{2}$$
 or $\theta=\frac{\pi}{6},\frac{5\pi}{6}$ Solve for θ in $[0,2\pi)$

Both sine and cosine have period 2π , so we get all the solutions of the equation by adding integer multiples of 2π to these solutions. Thus the solutions are

$$\theta = \frac{\pi}{2} + 2k\pi$$
 $\theta = \frac{3\pi}{2} + 2k\pi$ $\theta = \frac{\pi}{6} + 2k\pi$ $\theta = \frac{5\pi}{6} + 2k\pi$

where k is any integer.

NOW TRY EXERCISES 7 AND 11

EXAMPLE 3 | Squaring and Using an Identity

Solve the equation $\cos \theta + 1 = \sin \theta$ in the interval $[0, 2\pi)$.

SOLUTION To get an equation that involves either sine only or cosine only, we square both sides and use a Pythagorean identity:

$$\cos\theta + 1 = \sin\theta \qquad \qquad \text{Given equation}$$

$$\cos^2\theta + 2\cos\theta + 1 = \sin^2\theta \qquad \qquad \text{Square both sides}$$

$$\cos^2\theta + 2\cos\theta + 1 = 1 - \cos^2\theta \qquad \qquad \text{Pythagorean identity}$$

$$2\cos^2\theta + 2\cos\theta = 0 \qquad \qquad \text{Simplify}$$

$$2\cos\theta(\cos\theta + 1) = 0 \qquad \qquad \text{Factor}$$

$$2\cos\theta = 0 \qquad \text{or} \qquad \cos\theta + 1 = 0 \qquad \qquad \text{Set each factor equal to 0}$$

$$\cos\theta = 0 \qquad \text{or} \qquad \cos\theta = -1 \qquad \qquad \text{Solve for } \cos\theta$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \text{or} \qquad \theta = \pi \qquad \qquad \text{Solve for } \theta \text{ in } [0, 2\pi)$$

Because we squared both sides, we need to check for extraneous solutions. From *Check Your Answers* we see that the solutions of the given equation are $\pi/2$ and π .

CHECK YOUR ANSWER

$$\theta = \frac{\pi}{2}$$

$$\theta = \frac{3\pi}{2}$$

$$\theta = \pi$$

$$\cos \frac{\pi}{2} + 1 = \sin \frac{\pi}{2}$$

$$0 + 1 = 1$$

$$\theta = \pi$$

$$\cos \frac{3\pi}{2} + 1 = \sin \frac{3\pi}{2}$$

$$0 + 1 \stackrel{?}{=} -1 + 1 = 0$$

NOW TRY EXERCISE 13

EXAMPLE 4 | Finding Intersection Points

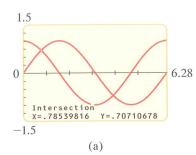
Find the values of x for which the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ intersect.

SOLUTION 1: Graphical

The graphs intersect where f(x)=g(x). In Figure 1 we graph $y_1=\sin x$ and $y_2=\cos x$ on the same screen, for x between 0 and 2π . Using TRACE or the intersect command on the graphing calculator, we see that the two points of intersection in this interval occur where $x\approx 0.785$ and $x\approx 3.927$. Since sine and cosine are periodic with period 2π , the intersection points occur where

$$x \approx 0.785 + 2k\pi$$
 and $x \approx 3.927 + 2k\pi$

where k is any integer.



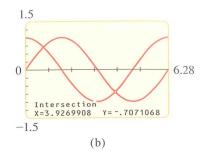


FIGURE 1

SOLUTION 2: Algebraic

To find the exact solution, we set f(x) = g(x) and solve the resulting equation algebraically:

$$\sin x = \cos x$$
 Equate functions

Since the numbers x for which $\cos x = 0$ are not solutions of the equation, we can divide both sides by $\cos x$:

$$\frac{\sin x}{\cos x} = 1$$
 Divide by $\cos x$
$$\tan x = 1$$
 Reciprocal identity

The only solution of this equation in the interval $(-\pi/2, \pi/2)$ is $x = \pi/4$. Since tangent has period π , we get all solutions of the equation by adding integer multiples of π :

$$x = \frac{\pi}{4} + k\pi$$

where k is any integer. The graphs intersect for these values of x. You should use your calculator to check that, rounded to three decimals, these are the same values that we obtained in Solution 1.

NOW TRY EXERCISE 35

▼ Equations with Trigonometric Functions of Multiples of Angles

When solving trigonometric equations that involve functions of multiples of angles, we first solve for the multiple of the angle, then divide to solve for the angle.

EXAMPLE 5 | A Trigonometric Equation Involving a Multiple of an Angle

Consider the equation $2 \sin 3\theta - 1 = 0$.

- (a) Find all solutions of the equation.
- (b) Find the solutions in the interval $[0, 2\pi)$.

(a) We first isolate $\sin 3\theta$ and then solve for the angle 3θ .

$$2 \sin 3\theta - 1 = 0$$
 Given equation
$$2 \sin 3\theta = 1$$
 Add 1
$$\sin 3\theta = \frac{1}{2}$$
 Divide by 2
$$3\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$
 Solve for 3θ in the interval $[0, 2\pi)$ (see Figure 2)

To get all solutions, we add integer multiples of 2π to these solutions. So the solutions are of the form

$$3\theta = \frac{\pi}{6} + 2k\pi \qquad 3\theta = \frac{5\pi}{6} + 2k\pi$$

To solve for θ , we divide by 3 to get the solutions

$$\theta = \frac{\pi}{18} + \frac{2k\pi}{3}$$
 $\theta = \frac{5\pi}{18} + \frac{2k\pi}{3}$

where k is any integer.

(b) The solutions from part (a) that are in the interval $[0, 2\pi)$ correspond to k = 0, 1, and 2. For all other values of k the corresponding values of θ lie outside this interval. So the solutions in the interval $[0, 2\pi)$ are

$$\theta = \underbrace{\frac{\pi}{18}, \frac{5\pi}{18}}_{k=0}, \underbrace{\frac{13\pi}{18}, \frac{17\pi}{18}}_{k=1}, \underbrace{\frac{25\pi}{18}, \frac{29\pi}{18}}_{k=2}$$

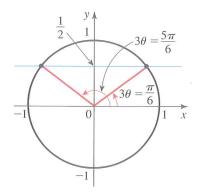


FIGURE 2

NOW TRY EXERCISE 17

EXAMPLE 6 | A Trigonometric Equation Involving a Half Angle

Consider the equation $\sqrt{3} \tan \frac{\theta}{2} - 1 = 0$.

- (a) Find all solutions of the equation.
- (b) Find the solutions in the interval $[0, 4\pi)$.

SOLUTION

(a) We start by isolating $\tan \frac{\theta}{2}$:

$$\sqrt{3} \tan \frac{\theta}{2} - 1 = 0$$
 Given equation
$$\sqrt{3} \tan \frac{\theta}{2} = 1$$
 Add 1
$$\tan \frac{\theta}{2} = \frac{1}{\sqrt{3}}$$
 Divide by $\sqrt{3}$
$$\frac{\theta}{2} = \frac{\pi}{6}$$
 Solve for $\frac{\theta}{2}$ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Since tangent has period π , to get all solutions we add integer multiples of π to this solution. So the solutions are of the form

$$\frac{\theta}{2} = \frac{\pi}{6} + k\pi$$

Multiplying by 2, we get the solutions

$$\theta = \frac{\pi}{3} + 2k\pi$$

where k is any integer.

(b) The solutions from part (a) that are in the interval $[0, 4\pi)$ correspond to k=0 and k = 1. For all other values of k the corresponding values of x lie outside this interval. Thus the solutions in the interval $[0, 4\pi)$ are

$$x = \frac{\pi}{3}, \, \frac{7\pi}{3}$$

NOW TRY EXERCISE 23

7.5 EXERCISES

CONCEPTS

1–2 ■ We can use identities to help us solve trigonometric equations.

- 1. Using a Pythagorean identity we see that the equation $\sin x + \sin^2 x + \cos^2 x = 1$ is equivalent to the basic equation whose solutions are x =
- 2. Using a Double-Angle Formula we see that the equation $\sin x + \sin 2x = 0$ is equivalent to the equation Factoring we see that solving this equation is equivalent to solving the two basic equations _____ and ___

SKILLS

3–16 ■ Solve the given equation.

3.
$$2\cos^2\theta + \sin\theta = 1$$
 4. $\sin^2\theta = 4 - 2\cos^2\theta$

4.
$$\sin^2 \theta = 4 - 2 \cos^2 \theta$$

5.
$$\tan^2\theta - 2\sec\theta = 2$$

$$6. \csc^2 \theta = \cot \theta + 3$$

$$\sim$$
 7. $2 \sin 2\theta - 3 \sin \theta = 0$

9.
$$\cos 2\theta = 3 \sin \theta - 1$$

• 11.
$$2 \sin^2 \theta - \cos \theta = 1$$

$$13. \sin \theta - 1 = \cos \theta$$

15.
$$\tan \theta + 1 = \sec \theta$$

$$8. 3 \sin 2\theta - 2 \sin \theta = 0$$

10.
$$\cos 2\theta = \cos^2 \theta - \frac{1}{2}$$

12.
$$\tan \theta - 3 \cot \theta = 0$$

14.
$$\cos \theta - \sin \theta = 1$$

16.
$$2 \tan \theta + \sec^2 \theta = 4$$

17–34 ■ An equation is given. (a) Find all solutions of the equation. (b) Find the solutions in the interval $[0, 2\pi)$.

• 17.
$$2\cos 3\theta = 1$$

18.
$$3 \csc^2 \theta = 4$$

19.
$$2\cos 2\theta + 1 = 0$$

20.
$$2 \sin 3\theta + 1 = 0$$

21.
$$\sqrt{3} \tan 3\theta + 1 = 0$$
 22. $\sec 4\theta - 2 = 0$

23.
$$\cos \frac{\theta}{2} - 1 = 0$$
 24. $\tan \frac{\theta}{4} + \sqrt{3} = 0$

24.
$$\tan \frac{\theta}{1} + \sqrt{3} = 0$$

25.
$$2 \sin \frac{\theta}{3} + \sqrt{3} = 0$$
 26. $\sec \frac{\theta}{2} = \cos \frac{\theta}{2}$

26.
$$\sec \frac{\theta}{2} = \cos \frac{\theta}{2}$$

27.
$$\sin 2\theta = 3 \cos 2\theta$$

28.
$$\csc 3\theta = 5 \sin 3\theta$$

29.
$$\sec \theta - \tan \theta = \cos \theta$$

30.
$$\tan 3\theta + 1 = \sec 3\theta$$

31.
$$3 \tan^3 \theta - 3 \tan^2 \theta - \tan \theta + 1 = 0$$

32.
$$4 \sin \theta \cos \theta + 2 \sin \theta - 2 \cos \theta - 1 = 0$$

33.
$$2 \sin \theta \tan \theta - \tan \theta = 1 - 2 \sin \theta$$

34.
$$\sec \theta \tan \theta - \cos \theta \cot \theta = \sin \theta$$

35–38 ■ (a) Graph f and g in the given viewing rectangle and find the intersection points graphically, rounded to two decimal places.

(b) Find the intersection points of f and g algebraically. Give exact

35.
$$f(x) = 3 \cos x + 1, g(x) = \cos x - 1;$$

 $[-2\pi, 2\pi]$ by $[-2.5, 4.5]$

36.
$$f(x) = \sin 2x + 1$$
, $g(x) = 2 \sin 2x + 1$; $[-2\pi, 2\pi]$ by $[-1.5, 3.5]$

37.
$$f(x) = \tan x, g(x) = \sqrt{3}; \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \text{ by } [-10, 10]$$

38.
$$f(x) = \sin x - 1$$
, $g(x) = \cos x$; $[-2\pi, 2\pi]$ by $[-2.5, 1.5]$

39–42 ■ Use an Addition or Subtraction Formula to simplify the equation. Then find all solutions in the interval $[0, 2\pi)$.

39.
$$\cos \theta \cos 3\theta - \sin \theta \sin 3\theta = 0$$

40.
$$\cos \theta \cos 2\theta + \sin \theta \sin 2\theta = \frac{1}{2}$$

41.
$$\sin 2\theta \cos \theta - \cos 2\theta \sin \theta = \sqrt{3/2}$$

42.
$$\sin 3\theta \cos \theta - \cos 3\theta \sin \theta = 0$$

43–52 ■ Use a Double- or Half-Angle Formula to solve the equation in the interval $[0, 2\pi)$.

$$43. \sin 2\theta + \cos \theta = 0$$

44.
$$\tan \frac{\theta}{2} - \sin \theta = 0$$

45.
$$\cos 2\theta + \cos \theta = 2$$

46.
$$\tan \theta + \cot \theta = 4 \sin 2\theta$$

$$47. \cos 2\theta - \cos^2 \theta = 0$$

48.
$$2 \sin^2 \theta = 2 + \cos 2\theta$$

$$49. \cos 2\theta - \cos 4\theta = 0$$

50.
$$\sin 3\theta - \sin 6\theta = 0$$

51.
$$\cos \theta - \sin \theta = \sqrt{2} \sin \frac{\theta}{2}$$
 52. $\sin \theta - \cos \theta = \frac{1}{2}$

52.
$$\sin \theta - \cos \theta = \frac{1}{2}$$

53–56 ■ Solve the equation by first using a Sum-to-Product Formula.

53.
$$\sin \theta + \sin 3\theta = 0$$

$$54. \cos 5\theta - \cos 7\theta = 0$$

55.
$$\cos 4\theta + \cos 2\theta = \cos \theta$$

56.
$$\sin 5\theta - \sin 3\theta = \cos 4\theta$$

57–62 ■ Use a graphing device to find the solutions of the equation, correct to two decimal places.

57.
$$\sin 2x = x$$

58.
$$\cos x = \frac{x}{2}$$

59.
$$2^{\sin x} = x$$

60.
$$\sin x = x^3$$

61.
$$\frac{\cos x}{1+x^2} = x^2$$

62.
$$\cos x = \frac{1}{2}(e^x + e^{-x})$$

APPLICATIONS

63. Range of a Projectile If a projectile is fired with velocity v_0 at an angle θ , then its range, the horizontal distance it travels (in feet), is modeled by the function

$$R(\theta) = \frac{v_0^2 \sin 2\theta}{32}$$

(See page 576.) If $v_0 = 2200$ ft/s, what angle (in degrees) should be chosen for the projectile to hit a target on the ground 5000 ft away?

64. Damped Vibrations The displacement of a spring vibrating in damped harmonic motion is given by

$$y = 4e^{-3t}\sin 2\pi t$$

Find the times when the spring is at its equilibrium position (y = 0).

65. Hours of Daylight In Philadelphia the number of hours of daylight on day t (where t is the number of days after January 1) is modeled by the function

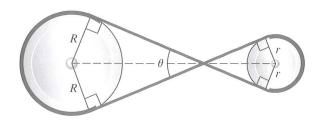
$$L(t) = 12 + 2.83 \sin\left(\frac{2\pi}{365}(t - 80)\right)$$

- (a) Which days of the year have about 10 hours of daylight?
- (b) How many days of the year have more than 10 hours of daylight?
- **66. Belts and Pulleys** A thin belt of length L surrounds two pulleys of radii R and r, as shown in the figure.
 - (a) Show that the angle θ (in radians) where the belt crosses itself satisfies the equation

$$\theta + 2\cot\frac{\theta}{2} = \frac{L}{R+r} - \pi$$

[Hint: Express L in terms of R, r, and θ by adding up the lengths of the curved and straight parts of the belt.]

(b) Suppose that R = 2.42 ft, r = 1.21 ft, and L = 27.78 ft. Find θ by solving the equation in part (a) graphically. Express your answer both in radians and in degrees.



DISCOVERY DISCUSSION WRITING

67. A Special Trigonometric Equation What makes the equation $\sin(\cos x) = 0$ different from all the other equations we've looked at in this section? Find all solutions of this equation.