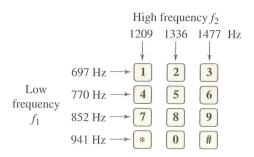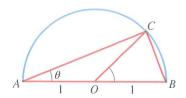**(c)** Graph $y = 2 \cos t$ and $y = -2 \cos t$, together with the graph in part (a), in the same viewing rectangle. How do these graphs describe the variation in the loudness of the sound?

108. Touch-Tone Telephones When a key is pressed on a touch-tone telephone, the keypad generates two pure tones, which combine to produce a sound that uniquely identifies the key. The figure shows the low frequency f_1 and the high frequency f_2 associated with each key. Pressing a key produces the sound wave $y = \sin(2\pi f_1 t) + \sin(2\pi f_2 t)$.

(a) Find the function that models the sound produced when the 4 key is pressed.

(b) Use a Sum-to-Product Formula to express the sound generated by the 4 key as a product of a sine and a cosine function.

(c) Graph the sound wave generated by the 4 key, from $t = 0$ to $t = 0.006$ s.

High frequency f_2

	1209	1336	1477 Hz
697 Hz →	1	2	3
770 Hz →	4	5	6
852 Hz →	7	8	9
941 Hz →	*	0	#

Low frequency f_1

DISCOVERY ▪ DISCUSSION ▪ WRITING

109. Geometric Proof of a Double-Angle Formula
Use the figure to prove that $\sin 2\theta = 2 \sin \theta \cos \theta$.

[*Hint:* Find the area of triangle ABC in two different ways. You will need the following facts from geometry:

An angle inscribed in a semicircle is a right angle, so $\angle ACB$ is a right angle.

The central angle subtended by the chord of a circle is twice the angle subtended by the chord on the circle, so $\angle BOC$ is 2θ.]

DISCOVERY PROJECT | **Where to Sit at the Movies**

In this project we use trigonometry to find the best location to observe such things as a painting or a movie. You can find the project at the book companion website:
www.stewartmath.com

7.4 BASIC TRIGONOMETRIC EQUATIONS

Basic Trigonometric Equations ▶ Solving Trigonometric Equations by Factoring

An equation that contains trigonometric functions is called a **trigonometric equation**. For example, the following are trigonometric equations:

$$\sin^2\theta + \cos^2\theta = 1 \qquad 2\sin\theta - 1 = 0 \qquad \tan 2\theta - 1 = 0$$

The first equation is an *identity*—that is, it is true for every value of the variable θ. The other two equations are true only for certain values of θ. To solve a trigonometric equation, we find all the values of the variable that make the equation true.

▼ Basic Trigonometric Equations

Solving any trigonometric equation always reduces to solving a **basic trigonometric equation**—an equation of the form $T(\theta) = c$, where T is a trigonometric function and c is a constant. In the next three examples we solve such basic equations.

EXAMPLE 1 | Solving a Basic Trigonometric Equation
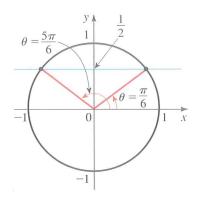
Solve the equation $\sin\theta = \dfrac{1}{2}$.

FIGURE 1

SOLUTION

Find the solutions in one period. Because sine has period 2π, we first find the solutions in any interval of length 2π. To find these solutions, we look at the unit circle in Figure 1. We see that $\sin \theta = \frac{1}{2}$ in Quadrants I and II, so the solutions in the interval $[0, 2\pi)$ are

$$\theta = \frac{\pi}{6} \qquad \theta = \frac{5\pi}{6}$$

Find all solutions. Because the sine function repeats its values every 2π units, we get all solutions of the equation by adding integer multiples of 2π to these solutions:

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi$$

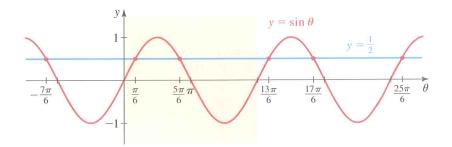where k is any integer. Figure 2 gives a graphical representation of the solutions.

FIGURE 2

✦ NOW TRY EXERCISE **5**

EXAMPLE 2 | Solving a Basic Trigonometric Equation

Solve the equation $\cos \theta = -\dfrac{\sqrt{2}}{2}$, and list eight specific solutions.

SOLUTION

Find the solutions in one period. Because cosine has period 2π, we first find the solutions in any interval of length 2π. From the unit circle in Figure 3 we see that $\cos \theta = -\sqrt{2}/2$ in Quadrants II and III, so the solutions in the interval $[0, 2\pi)$ are

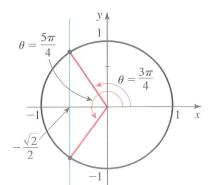$$\theta = \frac{3\pi}{4} \qquad \theta = \frac{5\pi}{4}$$

FIGURE 3

Find all solutions. Because the cosine function repeats its values every 2π units, we get all solutions of the equation by adding integer multiples of 2π to these solutions:

$$\theta = \frac{3\pi}{4} + 2k\pi \qquad \theta = \frac{5\pi}{4} + 2k\pi$$

where k is any integer. You can check that for $k = -1, 0, 1, 2$ we get the following specific solutions:

$$\theta = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{19\pi}{4}, \frac{21\pi}{4}$$

$$\underbrace{\qquad}_{k=-1} \quad \underbrace{\qquad}_{k=0} \quad \underbrace{\qquad}_{k=1} \quad \underbrace{\qquad}_{k=2}$$

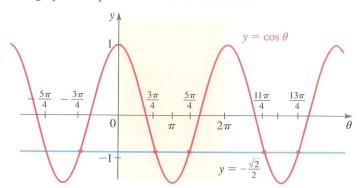Figure 4 gives a graphical representation of the solutions.

FIGURE 4

🖉 NOW TRY EXERCISE **17**

EXAMPLE 3 | Solving a Basic Trigonometric Equation

Solve the equation $\cos \theta = 0.65$.

SOLUTION

Find the solutions in one period.   We first find one solution by taking $\cos^{-1}$ of each side of the equation.

$$\cos \theta = 0.65 \qquad \text{Given equation}$$

$$\theta = \cos^{-1}(0.65) \qquad \text{Take } \cos^{-1} \text{ of each side}$$

$$\theta \approx 0.86 \qquad \text{Calculator (in radian mode)}$$

Because cosine has period 2π, we next find the solutions in any interval of length 2π. To find these solutions, we look at the unit circle in Figure 5. We see that $\cos \theta = 0.85$ in Quadrants I and IV, so the solutions are

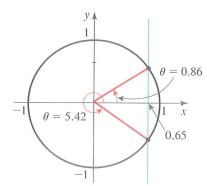$$\theta \approx 0.86 \qquad \theta \approx 2\pi - 0.86 \approx 5.42$$

FIGURE 5

Find all solutions.   To get all solutions of the equation, we add integer multiples of 2π to these solutions:

$$\theta \approx 0.86 + 2k\pi \qquad \theta \approx 5.42 + 2k\pi$$

where k is any integer.

🖉 NOW TRY EXERCISE **21**

EXAMPLE 4 | Solving a Basic Trigonometric Equation

Solve the equation $\tan \theta = 2$.

SOLUTION

Find the solutions in one period. We first find one solution by taking $\tan^{-1}$ of each side of the equation:

$$\tan \theta = 2 \qquad \text{Given equation}$$

$$\theta = \tan^{-1}(2) \qquad \text{Take } \tan^{-1} \text{ of each side}$$

$$\theta \approx 1.12 \qquad \text{Calculator (in radian mode)}$$

By the definition of $\tan^{-1}$, the solution that we obtained is the only solution in the interval $(-\pi/2, \pi/2)$ (which is an interval of length π).

Find all solutions. Since tangent has period π, we get all solutions of the equation by adding integer multiples of π:

$$\theta \approx 1.12 + k\pi$$

where k is any integer. A graphical representation of the solutions is shown in Figure 6. You can check that the solutions shown in the graph correspond to $k = -1, 0, 1, 2, 3$.
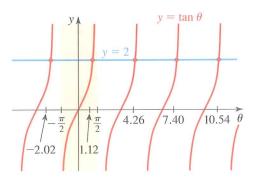

FIGURE 6

NOW TRY EXERCISE 23

In the next example we solve trigonometric equations that are algebraically equivalent to basic trigonometric equations.

EXAMPLE 5 | Solving Trigonometric Equations

Find all solutions of the equation.
(a) $2 \sin \theta - 1 = 0$ **(b)** $\tan^2 \theta - 3 = 0$

SOLUTION

(a) We start by isolating $\sin \theta$:

$$2 \sin \theta - 1 = 0 \qquad \text{Given equation}$$

$$2 \sin \theta = 1 \qquad \text{Add 1}$$

$$\sin \theta = \frac{1}{2} \qquad \text{Divide by 2}$$

This last equation is the same as that in Example 1. The solutions are

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi$$

where k is any integer.

(b) We start by isolating $\tan \theta$:

$$\tan^2 \theta - 3 = 0 \qquad \text{Given equation}$$

$$\tan^2 \theta = 3 \qquad \text{Add 3}$$

$$\tan \theta = \pm\sqrt{3} \qquad \text{Take the square root}$$

Because tangent has period π, we first find the solutions in any interval of length π. In the interval $(-\pi/2, \pi/2)$ the solutions are $\theta = \pi/3$ and $\theta = -\pi/3$. To get all solutions, we add integer multiples of π to these solutions:

$$\theta = \frac{\pi}{3} + k\pi \qquad \theta = -\frac{\pi}{3} + k\pi$$

where k is any integer.

. NOW TRY EXERCISES 27 AND 33

▼ Solving Trigonometric Equations by Factoring

Factoring is one of the most useful techniques for solving equations, including trigonometric equations. The idea is to move all terms to one side of the equation, factor, and then use the Zero-Product Property (see Section 1.5).

Zero-Product Property
If $AB = 0$, then $A = 0$ or $B = 0$.

EXAMPLE 6 | A Trigonometric Equation of Quadratic Type

Solve the equation $2\cos^2 \theta - 7\cos \theta + 3 = 0$.

SOLUTION   We factor the left-hand side of the equation.

Equation of Quadratic Type
$$2C^2 - 7C + 3 = 0$$
$$(2C - 1)(C - 3) = 0$$

$$2\cos^2 \theta - 7\cos \theta + 3 = 0 \qquad \text{Given equation}$$

$$(2\cos \theta - 1)(\cos \theta - 3) = 0 \qquad \text{Factor}$$

$$2\cos \theta - 1 = 0 \quad \text{or} \quad \cos \theta - 3 = 0 \qquad \text{Set each factor equal to 0}$$

$$\cos \theta = \frac{1}{2} \quad \text{or} \quad \cos \theta = 3 \qquad \text{Solve for } \cos \theta$$

Because cosine has period 2π, we first find the solutions in the interval $[0, 2\pi)$. For the first equation the solutions are $\theta = \pi/3$ and $\theta = 5\pi/3$ (see Figure 7). The second equation has no solution because $\cos \theta$ is never greater than 1. Thus the solutions are

$$\theta = \frac{\pi}{3} + 2k\pi \qquad \theta = \frac{5\pi}{3} + 2k\pi$$
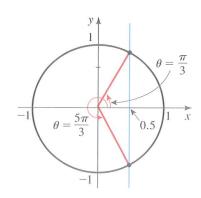
where k is any integer.

FIGURE 7

. NOW TRY EXERCISE 41

EXAMPLE 7 | Solving a Trigonometric Equation by Factoring

Solve the equation $5 \sin \theta \cos \theta + 4 \cos \theta = 0$.

SOLUTION We factor the left-hand side of the equation:

$$5 \sin \theta \cos \theta + 2 \cos \theta = 0 \qquad \text{Given equation}$$
$$\cos \theta (5 \sin \theta + 2) = 0 \qquad \text{Factor}$$
$$\cos \theta = 0 \quad \text{or} \quad 5 \sin \theta + 4 = 0 \qquad \text{Set each factor equal to 0}$$
$$\sin \theta = -0.8 \qquad \text{Solve for } \sin \theta$$

Because sine and cosine have period 2π, we first find the solutions of these equations in an interval of length 2π. For the first equation the solutions in the interval $[0, 2\pi)$ are $\theta = \pi/2$ and $\theta = 3\pi/2$. To solve the second equation, we take $\sin^{-1}$ of each side:

$$\sin \theta = -0.80 \qquad \text{Second equation}$$
$$\theta = \sin^{-1}(-0.80) \qquad \text{Take } \sin^{-1} \text{ of each side}$$
$$\theta \approx -0.93 \qquad \text{Calculator (in radian mode)}$$

So the solutions in an interval of length 2π are $\theta = -0.93$ and $\theta = \pi + 0.93 \approx 4.07$ (see Figure 8). We get all the solutions of the equation by adding integer multiples of 2π to these solutions.

$$\theta = \frac{\pi}{2} + 2k\pi, \qquad \theta = \frac{3\pi}{2} + 2k\pi, \qquad \theta \approx -0.93 + 2k\pi, \qquad \theta \approx 4.07 + 2k\pi$$

where k is any integer.

NOW TRY EXERCISE 53

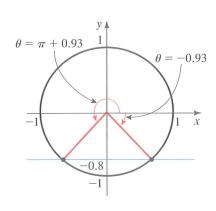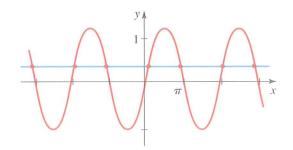$\theta = \pi + 0.93$
$\theta = -0.93$

FIGURE 8

7.4 EXERCISES

CONCEPTS

1. Because the trigonometric functions are periodic, if a basic trigonometric equation has one solution, it has _____ (several/infinitely many) solutions.

2. The basic equation $\sin x = 2$ has _____ (no/one/infinitely many) solutions, whereas the basic equation $\sin x = 0.3$ has _____ (no/one/infinitely many) solutions.

3. We can find some of the solutions of $\sin x = 0.3$ graphically by graphing $y = \sin x$ and $y = $ _____. Use the graph below to estimate some of the solutions.

4. We can find the solutions of $\sin x = 0.3$ algebraically.
   (a) First we find the solutions in the interval $[-\pi, \pi]$. We get one such solution by taking $\sin^{-1}$ to get $x = $ _____. The other solution in this interval is $x = $ _____.
   (b) We find all solutions by adding multiples of _____ to the solutions in $[-\pi, \pi]$. The solutions are $x = $ _____ and $x = $ _____.

SKILLS

5–16 ■ Solve the given equation.

5. $\sin \theta = \dfrac{\sqrt{3}}{2}$
6. $\sin \theta = -\dfrac{\sqrt{2}}{2}$
7. $\cos \theta = -1$
8. $\cos \theta = \dfrac{\sqrt{3}}{2}$
9. $\cos \theta = \frac{1}{4}$
10. $\sin \theta = -0.3$
11. $\sin \theta = -0.45$
12. $\cos \theta = 0.32$
13. $\tan \theta = -\sqrt{3}$
14. $\tan \theta = 1$
15. $\tan \theta = 5$
16. $\tan \theta = -\frac{1}{3}$

17–24 ■ Solve the given equation, and list six specific solutions.

17. $\cos \theta = -\dfrac{\sqrt{3}}{2}$

18. $\cos \theta = \dfrac{1}{2}$

19. $\sin \theta = \dfrac{\sqrt{2}}{2}$

20. $\sin \theta = -\dfrac{\sqrt{3}}{2}$

21. $\cos \theta = 0.28$

22. $\tan \theta = 2.5$

23. $\tan \theta = -10$

24. $\sin \theta = -0.9$

25–38 ■ Find all solutions of the given equation.

25. $\cos \theta + 1 = 0$

26. $\sin \theta + 1 = 0$

27. $\sqrt{2} \sin \theta + 1 = 0$

28. $\sqrt{2} \cos \theta - 1 = 0$

29. $5 \sin \theta - 1 = 0$

30. $4 \cos \theta + 1 = 0$

31. $3 \tan^2 \theta - 1 = 0$

32. $\cot \theta + 1 = 0$

33. $2 \cos^2 \theta - 1 = 0$

34. $4 \sin^2 \theta - 3 = 0$

35. $\tan^2 \theta - 4 = 0$

36. $9 \sin^2 \theta - 1 = 0$

37. $\sec^2 \theta - 2 = 0$

38. $\csc^2 \theta - 4 = 0$

39–56 ■ Solve the given equation.

39. $(\tan^2 \theta - 4)(2 \cos \theta + 1) = 0$

40. $(\tan \theta - 2)(16 \sin^2 \theta - 1) = 0$

41. $4 \cos^2 \theta - 4 \cos \theta + 1 = 0$

42. $2 \sin^2 \theta - \sin \theta - 1 = 0$

43. $3 \sin^2 \theta - 7 \sin \theta + 2 = 0$

44. $\tan^4 \theta - 13 \tan^2 \theta + 36 = 0$

45. $2 \cos^2 \theta - 7 \cos \theta + 3 = 0$

46. $\sin^2 \theta - \sin \theta - 2 = 0$

47. $\cos^2 \theta - \cos \theta - 6 = 0$

48. $2 \sin^2 \theta + 5 \sin \theta - 12 = 0$

49. $\sin^2 \theta = 2 \sin \theta + 3$

50. $3 \tan^3 \theta = \tan \theta$

51. $\cos \theta (2 \sin \theta + 1) = 0$

52. $\sec \theta (2 \cos \theta - \sqrt{2}) = 0$

53. $\cos \theta \sin \theta - 2 \cos \theta = 0$

54. $\tan \theta \sin \theta + \sin \theta = 0$

55. $3 \tan \theta \sin \theta - 2 \tan \theta = 0$

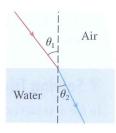**56.** $4 \cos \theta \sin \theta + 3 \cos \theta = 0$

APPLICATIONS

57. Refraction of Light It has been observed since ancient times that light refracts or "bends" as it travels from one medium to another (from air to water, for example). If v_1 is the speed of light in one medium and v_2 its speed in another medium, then according to **Snell's Law**,

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

where θ_1 is the *angle of incidence* and θ_2 is the *angle of refraction* (see the figure). The number v_1/v_2 is called the *index of refraction*. The index of refraction for several substances is given in the table.

If a ray of light passes through the surface of a lake at an angle of incidence of 70°, what is the angle of refraction?

Substance	Refraction from air to substance
Water	1.33
Alcohol	1.36
Glass	1.52
Diamond	2.41

58. Total Internal Reflection When light passes from a more-dense to a less-dense medium—from glass to air, for example—the angle of refraction predicted by Snell's Law (see Exercise 57) can be 90° or larger. In this case the light beam is actually reflected back into the denser medium. This phenomenon, called *total internal reflection*, is the principle behind fiber optics. Set $\theta_2 = 90°$ in Snell's Law, and solve for θ_1 to determine the critical angle of incidence at which total internal reflection begins to occur when light passes from glass to air. (Note that the index of refraction from glass to air is the reciprocal of the index from air to glass.)

59. Phases of the Moon As the moon revolves around the earth, the side that faces the earth is usually just partially illuminated by the sun. The phases of the moon describe how much of the surface appears to be in sunlight. An astronomical measure of phase is given by the fraction F of the lunar disc that is lit. When the angle between the sun, earth, and moon is θ ($0 \le \theta \le 360°$), then

$$F = \frac{1}{2}(1 - \cos \theta)$$

Determine the angles θ that correspond to the following phases:
(a) $F = 0$ (new moon)
(b) $F = 0.25$ (a crescent moon)
(c) $F = 0.5$ (first or last quarter)
(d) $F = 1$ (full moon)

DISCOVERY ■ DISCUSSION ■ WRITING

60. Equations and Identities Which of the following statements is true?

A. Every identity is an equation.

B. Every equation is an identity.

Give examples to illustrate your answer. Write a short paragraph to explain the difference between an equation and an identity.

7.5 MORE TRIGONOMETRIC EQUATIONS

Solving Trigonometric Equations by Using Identities ▶ Equations with Trigonometric Functions of Multiples of Angles

In this section we solve trigonometric equations by first using identities to simplify the equation. We also solve trigonometric equations in which the terms contain multiples of angles.

▼ Solving Trigonometric Equations by Using Identities

In the next two examples we use trigonometric identities to express a trigonometric equation in a form in which it can be factored.

EXAMPLE 1 | Using a Trigonometric Identity

Solve the equation $1 + \sin \theta = 2 \cos^2 \theta$.

SOLUTION We first need to rewrite this equation so that it contains only one trigonometric function. To do this, we use a trigonometric identity:

$$1 + \sin \theta = 2 \cos^2 \theta \qquad \text{Given equation}$$

$$1 + \sin \theta = 2(1 - \sin^2 \theta) \qquad \text{Pythagorean identity}$$

$$2 \sin^2 \theta + \sin \theta - 1 = 0 \qquad \text{Put all terms on one side}$$

$$(2 \sin \theta - 1)(\sin \theta + 1) = 0 \qquad \text{Factor}$$

$$2 \sin \theta - 1 = 0 \qquad \text{or} \qquad \sin \theta + 1 = 0 \qquad \text{Set each factor equal to 0}$$

$$\sin \theta = \frac{1}{2} \qquad \text{or} \qquad \sin \theta = -1 \qquad \text{Solve for } \sin \theta$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \qquad \text{or} \qquad \theta = \frac{3\pi}{2} \qquad \begin{array}{l}\text{Solve for } \theta \text{ in the} \\ \text{interval } [0, 2\pi)\end{array}$$

Because sine has period 2π, we get all the solutions of the equation by adding integer multiples of 2π to these solutions. Thus the solutions are

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi \qquad \theta = \frac{3\pi}{2} + 2k\pi$$

where k is any integer.

✎ NOW TRY EXERCISE 3 ∎

EXAMPLE 2 | Using a Trigonometric Identity

Solve the equation $\sin 2\theta - \cos \theta = 0$.

SOLUTION The first term is a function of 2θ, and the second is a function of θ, so we begin by using a trigonometric identity to rewrite the first term as a function of θ only:

$$\sin 2\theta - \cos \theta = 0 \qquad \text{Given equation}$$

$$2 \sin \theta \cos \theta - \cos \theta = 0 \qquad \text{Double-Angle Formula}$$

$$\cos \theta \, (2 \sin \theta - 1) = 0 \qquad \text{Factor}$$