

(c) Find the acute angle formed by the two lines

$$y = \frac{1}{3}x + 1$$
 and  $y = -\frac{1}{2}x - 3$ 

(d) Show that if two lines are perpendicular, then the slope of one is the negative reciprocal of the slope of the other. [*Hint*: First find an expression for  $\cot \psi$ .]



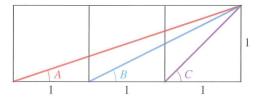
**65–66** ■ (a) Graph the function and make a conjecture, then

(b) prove that your conjecture is true.

**65.** 
$$y = \sin^2\left(x + \frac{\pi}{4}\right) + \sin^2\left(x - \frac{\pi}{4}\right)$$
  
**66.**  $y = -\frac{1}{2}[\cos(x + \pi) + \cos(x - \pi)]$ 

**66.** 
$$y = -\frac{1}{2}[\cos(x+\pi) + \cos(x-\pi)]$$

67. Find  $\angle A + \angle B + \angle C$  in the figure. [Hint: First use an addition formula to find tan(A + B).]



### APPLICATIONS



- **68. Adding an Echo** A digital delay device echoes an input signal by repeating it a fixed length of time after it is received. If such a device receives the pure note  $f_1(t) = 5 \sin t$  and echoes the pure note  $f_2(t) = 5 \cos t$ , then the combined sound is  $f(t) = f_1(t) + f_2(t)$ .
  - (a) Graph y = f(t) and observe that the graph has the form of a sine curve  $y = k \sin(t + \phi)$ .
  - (b) Find k and  $\phi$ .

**69. Interference** Two identical tuning forks are struck, one a fraction of a second after the other. The sounds produced are modeled by  $f_1(t) = C \sin \omega t$  and  $f_2(t) = C \sin(\omega t + \alpha)$ . The two sound waves interfere to produce a single sound modeled by the sum of these functions

$$f(t) = C \sin \omega t + C \sin(\omega t + \alpha)$$

- (a) Use the Addition Formula for Sine to show that f can be written in the form  $f(t) = A \sin \omega t + B \cos \omega t$ , where A and B are constants that depend on  $\alpha$ .
- (b) Suppose that C = 10 and  $\alpha = \pi/3$ . Find constants k and  $\phi$  so that  $f(t) = k \sin(\omega t + \phi)$ .



### DISCOVERY - DISCUSSION - WRITING

**70. Addition Formula for Sine** In the text we proved only the Addition and Subtraction Formulas for Cosine. Use these formulas and the cofunction identities

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

to prove the Addition Formula for Sine. [Hint: To get started, use the first cofunction identity to write

$$\sin(s+t) = \cos\left(\frac{\pi}{2} - (s+t)\right)$$
$$= \cos\left(\left(\frac{\pi}{2} - s\right) - t\right)$$

and use the Subtraction Formula for Cosine.]

71. Addition Formula for Tangent Use the Addition Formulas for Cosine and Sine to prove the Addition Formula for Tangent. [Hint: Use

$$\tan(s+t) = \frac{\sin(s+t)}{\cos(s+t)}$$

and divide the numerator and denominator by  $\cos s \cos t$ .]

# Double-Angle, Half-Angle, and Product-Sum Formulas

Double-Angle Formulas ► Half-Angle Formulas ► Simplifying Expressions Involving Inverse Trigonometric Functions ► Product-Sum Formulas

The identities we consider in this section are consequences of the addition formulas. The **Double-Angle Formulas** allow us to find the values of the trigonometric functions at 2xfrom their values at x. The Half-Angle Formulas relate the values of the trigonometric functions at  $\frac{1}{2}x$  to their values at x. The **Product-Sum Formulas** relate products of sines and cosines to sums of sines and cosines.

## **▼ Double-Angle Formulas**

The formulas in the following box are immediate consequences of the addition formulas, which we proved in the preceding section.

### **DOUBLE-ANGLE FORMULAS**

Formula for sine:  $\sin 2x = 2 \sin x \cos x$ 

Formulas for cosine:  $\cos 2x = \cos^2 x - \sin^2 x$ 

$$= 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$

Formula for tangent:  $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$ 

The proofs for the formulas for cosine are given here. You are asked to prove the remaining formulas in Exercises 35 and 36.

#### PROOF OF DOUBLE-ANGLE FORMULAS FOR COSINE

$$\cos 2x = \cos(x + x)$$

$$= \cos x \cos x - \sin x \sin x$$

$$= \cos^2 x - \sin^2 x$$

The second and third formulas for  $\cos 2x$  are obtained from the formula we just proved and the Pythagorean identity. Substituting  $\cos^2 x = 1 - \sin^2 x$  gives

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= (1 - \sin^2 x) - \sin^2 x$$
$$= 1 - 2\sin^2 x$$

The third formula is obtained in the same way, by substituting  $\sin^2 x = 1 - \cos^2 x$ .

## **EXAMPLE 1** Using the Double-Angle Formulas

If  $\cos x = -\frac{2}{3}$  and x is in Quadrant II, find  $\cos 2x$  and  $\sin 2x$ .

**SOLUTION** Using one of the Double-Angle Formulas for Cosine, we get

$$\cos 2x = 2\cos^2 x - 1$$
$$= 2\left(-\frac{2}{3}\right)^2 - 1 = \frac{8}{9} - 1 = -\frac{1}{9}$$

To use the formula  $\sin 2x = 2 \sin x \cos x$ , we need to find  $\sin x$  first. We have

$$\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - \left(-\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$$

where we have used the positive square root because  $\sin x$  is positive in Quadrant II. Thus

$$\sin 2x = 2\sin x \cos x$$
$$= 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right) = -\frac{4\sqrt{5}}{9}$$

### **EXAMPLE 2** | A Triple-Angle Formula

Write  $\cos 3x$  in terms of  $\cos x$ .

### SOLUTION

$$\cos 3x = \cos(2x + x)$$

$$= \cos 2x \cos x - \sin 2x \sin x$$
Addition formula
$$= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x$$
Double-Angle Formulas
$$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$$
Expand
$$= 2\cos^3 x - \cos x - 2\cos x (1 - \cos^2 x)$$
Pythagorean identity
$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$
Expand
$$= 4\cos^3 x - 3\cos x$$
Simplify

### NOW TRY EXERCISE 101

Example 2 shows that  $\cos 3x$  can be written as a polynomial of degree 3 in  $\cos x$ . The identity  $\cos 2x = 2\cos^2 x - 1$  shows that  $\cos 2x$  is a polynomial of degree 2 in  $\cos x$ . In fact, for any natural number n, we can write  $\cos nx$  as a polynomial in  $\cos x$  of degree n (see the note following Exercise 101). The analogous result for  $\sin nx$  is not true in general.

## **EXAMPLE 3** | Proving an Identity

Prove the identity 
$$\frac{\sin 3x}{\sin x \cos x} = 4 \cos x - \sec x$$
.

### **SOLUTION** We start with the left-hand side:

$$\frac{\sin 3x}{\sin x \cos x} = \frac{\sin(x + 2x)}{\sin x \cos x}$$

$$= \frac{\sin x \cos 2x + \cos x \sin 2x}{\sin x \cos x}$$
Addition Formula
$$= \frac{\sin x (2 \cos^2 x - 1) + \cos x (2 \sin x \cos x)}{\sin x \cos x}$$
Double-Angle Formulas
$$= \frac{\sin x (2 \cos^2 x - 1)}{\sin x \cos x} + \frac{\cos x (2 \sin x \cos x)}{\sin x \cos x}$$
Separate fraction
$$= \frac{2 \cos^2 x - 1}{\cos x} + 2 \cos x$$
Cancel
$$= 2 \cos x - \frac{1}{\cos x} + 2 \cos x$$
Separate fraction
$$= 4 \cos x - \sec x$$
Reciprocal identity

# **▼** Half-Angle Formulas

NOW TRY EXERCISE 81

The following formulas allow us to write any trigonometric expression involving even powers of sine and cosine in terms of the first power of cosine only. This technique is important in calculus. The Half-Angle Formulas are immediate consequences of these formulas.

### FORMULAS FOR LOWERING POWERS

$$\sin^{2} x = \frac{1 - \cos 2x}{2}$$

$$\cos^{2} x = \frac{1 + \cos 2x}{2}$$

$$\tan^{2} x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

**PROOF** The first formula is obtained by solving for  $\sin^2 x$  in the double-angle formula  $\cos 2x = 1 - 2 \sin^2 x$ . Similarly, the second formula is obtained by solving for  $\cos^2 x$  in the Double-Angle Formula  $\cos 2x = 2 \cos^2 x - 1$ .

The last formula follows from the first two and the reciprocal identities:

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{\frac{1 - \cos 2x}{2}}{\frac{1 + \cos 2x}{2}} = \frac{1 - \cos 2x}{1 + \cos 2x}$$

## **EXAMPLE 4** | Lowering Powers in a Trigonometric Expression

Express  $\sin^2 x \cos^2 x$  in terms of the first power of cosine.

**SOLUTION** We use the formulas for lowering powers repeatedly:

$$\sin^2 x \cos^2 x = \left(\frac{1 - \cos 2x}{2}\right) \left(\frac{1 + \cos 2x}{2}\right)$$

$$= \frac{1 - \cos^2 2x}{4} = \frac{1}{4} - \frac{1}{4}\cos^2 2x$$

$$= \frac{1}{4} - \frac{1}{4} \left(\frac{1 + \cos 4x}{2}\right) = \frac{1}{4} - \frac{1}{8} - \frac{\cos 4x}{8}$$

$$= \frac{1}{8} - \frac{1}{8}\cos 4x = \frac{1}{8}(1 - \cos 4x)$$

Another way to obtain this identity is to use the Double-Angle Formula for Sine in the form  $\sin x \cos x = \frac{1}{2} \sin 2x$ . Thus

$$\sin^2 x \cos^2 x = \frac{1}{4} \sin^2 2x = \frac{1}{4} \left( \frac{1 - \cos 4x}{2} \right)$$
$$= \frac{1}{8} (1 - \cos 4x)$$

### NOW TRY EXERCISE 11

### HALF-ANGLE FORMULAS

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$
$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The choice of the + or - sign depends on the quadrant in which u/2 lies.

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**PROOF** We substitute x = u/2 in the formulas for lowering powers and take the square root of each side. This gives the first two Half-Angle Formulas. In the case of the Half-Angle Formula for Tangent we get

$$\tan \frac{u}{2} = \pm \sqrt{\frac{1 - \cos u}{1 + \cos u}}$$

$$= \pm \sqrt{\left(\frac{1 - \cos u}{1 + \cos u}\right) \left(\frac{1 - \cos u}{1 - \cos u}\right)} \qquad \text{Multiply numerator and denominator by } 1 - \cos u$$

$$= \pm \sqrt{\frac{(1 - \cos u)^2}{1 - \cos^2 u}} \qquad \text{Simplify}$$

$$= \pm \frac{|1 - \cos u|}{|\sin u|} \qquad \sqrt{A^2} = |A|$$
and  $1 - \cos^2 u = \sin^2 u$ 

Now,  $1 - \cos u$  is nonnegative for all values of u. It is also true that  $\sin u$  and  $\tan(u/2)$  always have the same sign. (Verify this.) It follows that

$$\tan\frac{u}{2} = \frac{1 - \cos u}{\sin u}$$

The other Half-Angle Formula for Tangent is derived from this by multiplying the numerator and denominator by  $1 + \cos u$ .

### **EXAMPLE 5** Using a Half-Angle Formula

Find the exact value of sin 22.5°.

**SOLUTION** Since 22.5° is half of 45°, we use the Half-Angle Formula for Sine with  $u = 45^{\circ}$ . We choose the + sign because 22.5° is in the first quadrant:

$$\sin \frac{45^{\circ}}{2} = \sqrt{\frac{1 - \cos 45^{\circ}}{2}} \qquad \text{Half-Angle Formula}$$

$$= \sqrt{\frac{1 - \sqrt{2}/2}{2}} \qquad \cos 45^{\circ} = \sqrt{2}/2$$

$$= \sqrt{\frac{2 - \sqrt{2}}{4}} \qquad \text{Common denominator}$$

$$= \frac{1}{2}\sqrt{2 - \sqrt{2}} \qquad \text{Simplify}$$

### NOW TRY EXERCISE 17

## **EXAMPLE 6** Using a Half-Angle Formula

Find tan(u/2) if  $sin u = \frac{2}{5}$  and u is in Quadrant II.

**50LUTION** To use the Half-Angle Formula for Tangent, we first need to find cos u. Since cosine is negative in Quadrant II, we have

$$\cos u = -\sqrt{1 - \sin^2 u}$$

$$= -\sqrt{1 - (\frac{2}{5})^2} = -\frac{\sqrt{21}}{5}$$

$$\tan \frac{u}{2} = \frac{1 - \cos u}{\sin u}$$

$$= \frac{1 + \sqrt{21/5}}{\frac{2}{5}} = \frac{5 + \sqrt{21}}{2}$$

Thus

 $\sqrt{1-x^2}$ 

## Evaluating Expressions Involving Inverse Trigonometric Functions

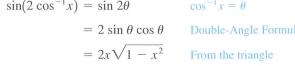
Expressions involving trigonometric functions and their inverses arise in calculus. In the next examples we illustrate how to evaluate such expressions.

#### EXAMPLE 7 Simplifying an Expression Involving an Inverse Trigonometric Function

Write  $\sin(2\cos^{-1}x)$  as an algebraic expression in x only, where  $-1 \le x \le 1$ .

**SOLUTION** Let  $\theta = \cos^{-1} x$ , and sketch a triangle as in Figure 1. We need to find  $\sin 2\theta$ , but from the triangle we can find trigonometric functions of  $\theta$  only, not  $2\theta$ . So we use the Double-Angle Formula for Sine.

$$\sin(2\cos^{-1}x) = \sin 2\theta$$
  $\cos^{-1}x = \theta$   
=  $2\sin \theta \cos \theta$  Double-Angle Formula  
=  $2x\sqrt{1-x^2}$  From the triangle



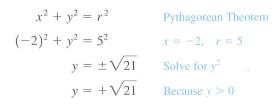


### NOW TRY EXERCISES 43 AND 47

### Evaluating an Expression Involving Inverse **EXAMPLE 8** Trigonometric Functions

Evaluate  $\sin 2\theta$ , where  $\cos \theta = -\frac{2}{5}$  with  $\theta$  in Quadrant II.

**SOLUTION** We first sketch the angle  $\theta$  in standard position with terminal side in Quadrant II as in Figure 2. Since  $\cos \theta = x/r = -\frac{2}{5}$ , we can label a side and the hypotenuse of the triangle in Figure 2. To find the remaining side, we use the Pythagorean Theorem:



We can now use the Double-Angle Formula for Sine:

$$\sin 2\theta = 2 \sin \theta \cos \theta$$
 Double-Angle Formula
$$= 2\left(\frac{\sqrt{21}}{5}\right)\left(-\frac{2}{5}\right)$$
 From the triangle
$$= -\frac{4\sqrt{21}}{25}$$
 Simplify

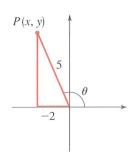


FIGURE 2

### NOW TRY EXERCISE 51

## Product-Sum Formulas

It is possible to write the product  $\sin u \cos v$  as a sum of trigonometric functions. To see this, consider the addition and subtraction formulas for the sine function:

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$
  
$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

Adding the left- and right-hand sides of these formulas gives

$$\sin(u+v) + \sin(u-v) = 2\sin u \cos v$$

Dividing by 2 gives the formula

$$\sin u \cos v = \frac{1}{2} \left[ \sin(u+v) + \sin(u-v) \right]$$

The other three **Product-to-Sum Formulas** follow from the addition formulas in a similar way.

### PRODUCT-TO-SUM FORMULAS

$$\sin u \cos v = \frac{1}{2} [\sin(u+v) + \sin(u-v)]$$

$$\cos u \sin v = \frac{1}{2} [\sin(u+v) - \sin(u-v)]$$

$$\cos u \cos v = \frac{1}{2} [\cos(u+v) + \cos(u-v)]$$

$$\sin u \sin v = \frac{1}{2} [\cos(u-v) - \cos(u+v)]$$

## **EXAMPLE 9** | Expressing a Trigonometric Product as a Sum

Express  $\sin 3x \sin 5x$  as a sum of trigonometric functions.

**SOLUTION** Using the fourth Product-to-Sum Formula with u = 3x and v = 5x and the fact that cosine is an even function, we get

$$\sin 3x \sin 5x = \frac{1}{2} [\cos(3x - 5x) - \cos(3x + 5x)]$$

$$= \frac{1}{2} \cos(-2x) - \frac{1}{2} \cos 8x$$

$$= \frac{1}{2} \cos 2x - \frac{1}{2} \cos 8x$$

### NOW TRY EXERCISE 55

The Product-to-Sum Formulas can also be used as Sum-to-Product Formulas. This is possible because the right-hand side of each Product-to-Sum Formula is a sum and the left side is a product. For example, if we let

$$u = \frac{x+y}{2}$$
 and  $v = \frac{x-y}{2}$ 

in the first Product-to-Sum Formula, we get

$$\sin\frac{x+y}{2}\cos\frac{x-y}{2} = \frac{1}{2}(\sin x + \sin y)$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

The remaining three of the following **Sum-to-Product Formulas** are obtained in a similar manner.

#### SUM-TO-PRODUCT FORMULAS

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

## **EXAMPLE 10** Expressing a Trigonometric Sum as a Product

Write  $\sin 7x + \sin 3x$  as a product.

**SOLUTION** The first Sum-to-Product Formula gives

$$\sin 7x + \sin 3x = 2\sin \frac{7x + 3x}{2}\cos \frac{7x - 3x}{2}$$

 $= 2 \sin 5x \cos 2x$ 

### NOW TRY EXERCISE 61

## **EXAMPLE 11** Proving an Identity

Verify the identity  $\frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$ .

**SOLUTION** We apply the second Sum-to-Product Formula to the numerator and the third formula to the denominator:

LHS = 
$$\frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \frac{2\cos\frac{3x + x}{2}\sin\frac{3x - x}{2}}{2\cos\frac{3x + x}{2}\cos\frac{3x - x}{2}}$$
Sum-to-Product Formulas

$$=\frac{2\cos 2x\sin x}{2\cos 2x\cos x}$$

Simplify

$$= \frac{\sin x}{\cos x} = \tan x = RHS$$

### NOW TRY EXERCISE 89

## 7.3 EXERCISES

### CONCEPTS

- 1. If we know the values of  $\sin x$  and  $\cos x$ , we can find the value of sin 2x by using the \_\_\_\_\_ Formula for Sine. State the formula:  $\sin 2x =$
- 2. If we know the value of  $\cos x$  and the quadrant in which x/2lies, we can find the value of sin(x/2) by using the \_ Formula for Sine. State the formula:

$$\sin(x/2) =$$
\_\_\_\_\_\_

### SKILLS

3–10 ■ Find  $\sin 2x$ ,  $\cos 2x$ , and  $\tan 2x$  from the given information.

3. 
$$\sin x = \frac{5}{13}$$
, x in Quadrant I

4. 
$$\tan x = -\frac{4}{3}$$
, x in Quadrant II

**5.** 
$$\cos x = \frac{4}{5}$$
,  $\csc x < 0$  **6.**  $\csc x = 4$ ,  $\tan x < 0$ 

6. 
$$\csc x = 4$$
,  $\tan x < 0$ 

7. 
$$\sin x = -\frac{3}{5}$$
, x in Quadrant III

**8.** 
$$\sec x = 2$$
,  $x$  in Quadrant IV

9. 
$$\tan x = -\frac{1}{3}$$
,  $\cos x > 0$ 

**10.** 
$$\cot x = \frac{2}{3}$$
,  $\sin x > 0$ 

11–16 ■ Use the formulas for lowering powers to rewrite the expression in terms of the first power of cosine, as in Example 4.

$$\sim$$
 11.  $\sin^4 x$ 

**12.** 
$$\cos^4 x$$

13. 
$$\cos^2 x \sin^4 x$$

**14.** 
$$\cos^4 x \sin^2 x$$

15. 
$$\cos^4 x \sin^4 x$$

**16.** 
$$\cos^6 x$$

17–28 ■ Use an appropriate Half-Angle Formula to find the exact value of the expression.

**23.** 
$$\tan \frac{\pi}{8}$$

**24.** 
$$\cos \frac{3\pi}{8}$$

**25.** 
$$\cos \frac{\pi}{12}$$

**26.** 
$$\tan \frac{5\pi}{12}$$

**27.** 
$$\sin \frac{9\pi}{8}$$

28. 
$$\sin \frac{11\pi}{12}$$

29–34 ■ Simplify the expression by using a Double-Angle Formula or a Half-Angle Formula.

**(b)** 
$$2 \sin 3\theta \cos 3\theta$$

30. (a) 
$$\frac{2 \tan 7^{\circ}}{1 - \tan^2 7^{\circ}}$$

**(b)** 
$$\frac{2 \tan 7\theta}{1 - \tan^2 7\theta}$$

31. (a) 
$$\cos^2 34^\circ - \sin^2 34^\circ$$

**(b)** 
$$\cos^2 5\theta - \sin^2 5\theta$$

**32.** (a) 
$$\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}$$
 (b)  $2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$ 

**(b)** 
$$2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

33. (a) 
$$\frac{\sin 8^{\circ}}{1 + \cos 8^{\circ}}$$

**(b)** 
$$\frac{1-\cos 4\theta}{\sin 4\theta}$$

34. (a) 
$$\sqrt{\frac{1-\cos 30^{\circ}}{2}}$$

**(b)** 
$$\sqrt{\frac{1-\cos 8\theta}{2}}$$

35. Use the Addition Formula for Sine to prove the Double-Angle Formula for Sine.

36. Use the Addition Formula for Tangent to prove the Double-Angle Formula for Tangent.

37-42 Find  $\sin \frac{x}{2}$ ,  $\cos \frac{x}{2}$ , and  $\tan \frac{x}{2}$  from the given information.

-37. 
$$\sin x = \frac{3}{5}$$
,  $0^{\circ} < x < 90^{\circ}$ 

38. 
$$\cos x = -\frac{4}{5}$$
,  $180^{\circ} < x < 270^{\circ}$ 

**39.** 
$$\csc x = 3$$
,  $90^{\circ} < x < 180^{\circ}$ 

**40.** 
$$\tan x = 1$$
,  $0^{\circ} < x < 90^{\circ}$ 

**41.** 
$$\sec x = \frac{3}{2}$$
,  $270^{\circ} < x < 360^{\circ}$ 

**42.** 
$$\cot x = 5$$
,  $180^{\circ} < x < 270^{\circ}$ 

**43–46** ■ Write the given expression as an algebraic expression in x.

$$^{\bullet}$$
 . 43.  $\sin(2 \tan^{-1} x)$ 

**44.** 
$$\tan(2\cos^{-1}x)$$

**45.** 
$$\sin(\frac{1}{2}\cos^{-1}x)$$

**46.** 
$$\cos(2\sin^{-1}x)$$

**47–50** ■ Find the exact value of the given expression.

$$^{*}$$
 . 47.  $\sin(2\cos^{-1}\frac{7}{25})$ 

**48.** 
$$\cos(2 \tan^{-1} \frac{12}{5})$$

**49.** 
$$\sec(2\sin^{-1}\frac{1}{4})$$

**50.** 
$$\tan(\frac{1}{2}\cos^{-1}\frac{2}{3})$$

51–54 ■ Evaluate each expression under the given conditions.

• 51. 
$$\cos 2\theta$$
;  $\sin \theta = -\frac{3}{5}$ ,  $\theta$  in Quadrant III

**52.** 
$$\sin(\theta/2)$$
;  $\tan \theta = -\frac{5}{12}$ ,  $\theta$  in Quadrant IV

53. 
$$\sin 2\theta$$
;  $\sin \theta = \frac{1}{7}$ ,  $\theta$  in Quadrant II

**54.** tan 
$$2\theta$$
; cos  $\theta = \frac{3}{5}$ ,  $\theta$  in Quadrant I

55-60 ■ Write the product as a sum.

**55.** 
$$\sin 2x \cos 3x$$

**56.** 
$$\sin x \sin 5x$$

57. 
$$\cos x \sin 4x$$

**58.** 
$$\cos 5x \cos 3x$$

**60.** 
$$11 \sin \frac{x}{2} \cos \frac{x}{4}$$

**61–66** ■ Write the sum as a product.

$$^{*}$$
 . **61.**  $\sin 5x + \sin 3x$ 

**62.** 
$$\sin x - \sin 4x$$

**63.** 
$$\cos 4x - \cos 6x$$

**64.** 
$$\cos 9x + \cos 2x$$

**65.** 
$$\sin 2x - \sin 7x$$

**66.** 
$$\sin 3x + \sin 4x$$

67–72 ■ Find the value of the product or sum.

**70.** 
$$\sin 75^{\circ} + \sin 15^{\circ}$$

71. 
$$\cos 255^{\circ} - \cos 195^{\circ}$$

72. 
$$\cos \frac{\pi}{12} + \cos \frac{5\pi}{12}$$

73-90 ■ Prove the identity.

73. 
$$\cos^2 5x - \sin^2 5x = \cos 10x$$

**74.** 
$$\sin 8x = 2 \sin 4x \cos 4x$$

**75.** 
$$(\sin x + \cos x)^2 = 1 + \sin 2x$$

**76.** 
$$\frac{2 \tan x}{1 + \tan^2 x} = \sin 2x$$

77. 
$$\frac{\sin 4x}{\sin x} = 4\cos x \cos 2x$$

78. 
$$\frac{1 + \sin 2x}{\sin 2x} = 1 + \frac{1}{2} \sec x \csc x$$

79. 
$$\frac{2(\tan x - \cot x)}{\tan^2 x - \cot^2 x} = \sin 2x$$

**80.** 
$$\cot 2x = \frac{1 - \tan^2 x}{2 \tan x}$$

**81.** 
$$\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$

**82.** 
$$4(\sin^6 x + \cos^6 x) = 4 - 3\sin^2 2x$$

83. 
$$\cos^4 x - \sin^4 x = \cos 2x$$

**84.** 
$$\tan^2\left(\frac{x}{2} + \frac{\pi}{4}\right) = \frac{1 + \sin x}{1 - \sin x}$$

**85.** 
$$\frac{\sin x + \sin 5x}{\cos x + \cos 5x} = \tan 3x$$

**86.** 
$$\frac{\sin 3x + \sin 7x}{\cos 3x - \cos 7x} = \cot 2x$$

87. 
$$\frac{\sin 10x}{\sin 9x + \sin x} = \frac{\cos 5x}{\cos 4x}$$

88. 
$$\frac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = \tan 3x$$

$$89. \frac{\sin x + \sin y}{\cos x + \cos y} = \tan \left( \frac{x + y}{2} \right)$$

**90.** 
$$\tan y = \frac{\sin(x+y) - \sin(x-y)}{\cos(x+y) + \cos(x-y)}$$

**91.** Show that 
$$\sin 130^{\circ} - \sin 110^{\circ} = -\sin 10^{\circ}$$
.

**92.** Show that 
$$\cos 100^{\circ} - \cos 200^{\circ} = \sin 50^{\circ}$$
.

**93.** Show that 
$$\sin 45^{\circ} + \sin 15^{\circ} = \sin 75^{\circ}$$
.

**94.** Show that 
$$\cos 87^{\circ} + \cos 33^{\circ} = \sin 63^{\circ}$$
.

95. Prove the identity

$$\frac{\sin x + \sin 2x + \sin 3x + \sin 4x + \sin 5x}{\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x} = \tan 3x$$

96. Use the identity

$$\sin 2x = 2 \sin x \cos x$$

n times to show that

 $\sin(2^n x) = 2^n \sin x \cos x \cos 2x \cos 4x \cdots \cos 2^{n-1} x$ 



- 97. (a) Graph  $f(x) = \frac{\sin 3x}{\sin x} \frac{\cos 3x}{\cos x}$  and make a conjecture.
  - (b) Prove the conjecture you made in part (a).



- 98. (a) Graph  $f(x) = \cos 2x + 2\sin^2 x$  and make a conjecture.
  - (b) Prove the conjecture you made in part (a).



- **99.** Let  $f(x) = \sin 6x + \sin 7x$ .
  - (a) Graph y = f(x).
  - (b) Verify that  $f(x) = 2 \cos \frac{1}{2}x \sin \frac{13}{2}x$ .
  - (c) Graph  $y = 2 \cos \frac{1}{2}x$  and  $y = -2 \cos \frac{1}{2}x$ , together with the graph in part (a), in the same viewing rectangle. How are these graphs related to the graph of f?
- 100. Let  $3x = \pi/3$  and let  $y = \cos x$ . Use the result of Example 2 to show that y satisfies the equation

$$8y^3 - 6y - 1 = 0$$

NOTE This equation has roots of a certain kind that are used to show that the angle  $\pi/3$  cannot be trisected by using a ruler and compass only.

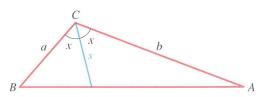
- 101. (a) Show that there is a polynomial P(t) of degree 4 such that  $\cos 4x = P(\cos x)$  (see Example 2).
  - (b) Show that there is a polynomial Q(t) of degree 5 such that  $\cos 5x = Q(\cos x)$ .

NOTE In general, there is a polynomial  $P_n(t)$  of degree n such that  $\cos nx = P_n(\cos x)$ . These polynomials are called Tchebycheff polynomials, after the Russian mathematician P. L. Tchebycheff (1821-1894).

**102.** In triangle ABC (see the figure) the line segment s bisects angle C. Show that the length of s is given by

$$s = \frac{2ab \cos x}{a+b}$$

[Hint: Use the Law of Sines.]

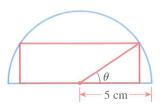


- **103.** If A, B, and C are the angles in a triangle, show that  $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
- 104. A rectangle is to be inscribed in a semicircle of radius 5 cm as shown in the following figure.
  - (a) Show that the area of the rectangle is modeled by the function

$$A(\theta) = 25 \sin 2\theta$$

(b) Find the largest possible area for such an inscribed rectangle.

(c) Find the dimensions of the inscribed rectangle with the largest possible area.



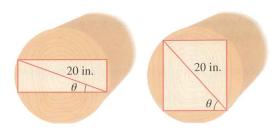
### APPLICATIONS

- 105. Sawing a Wooden Beam A rectangular beam is to be cut from a cylindrical log of diameter 20 in.
  - (a) Show that the cross-sectional area of the beam is modeled by the function

$$A(\theta) = 200 \sin 2\theta$$

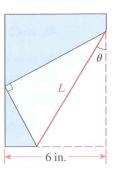
where  $\theta$  is as shown in the figure.

(b) Show that the maximum cross-sectional area of such a beam is 200 in<sup>2</sup>. [Hint: Use the fact that  $\sin u$  achieves its maximum value at  $u = \pi/2$ .]



**106.** Length of a Fold The lower right-hand corner of a long piece of paper 6 in. wide is folded over to the left-hand edge as shown. The length L of the fold depends on the angle  $\theta$ . Show that

$$L = \frac{3}{\sin\theta\cos^2\theta}$$



- **107. Sound Beats** When two pure notes that are close in frequency are played together, their sounds interfere to produce beats; that is, the loudness (or amplitude) of the sound alternately increases and decreases. If the two notes are given by

$$f_1(t) = \cos 11t$$
 and  $f_2(t) = \cos 13t$ 

the resulting sound is  $f(t) = f_1(t) + f_2(t)$ .

- (a) Graph the function y = f(t).
- **(b)** Verify that  $f(t) = 2 \cos t \cos 12t$ .