© Sony Pictures Classics/Courtesy of Everett Collection

CHAPTER 7

Analytic Trigonometry

In Chapters 5 and 6 we studied graphical and geometric properties of the trigonometric functions. In this chapter we study algebraic properties of these functions, that is, simplifying and factoring expressions and solving equations that involve trigonometric functions.

We have used the trigonometric functions to model different real-world phenomena, including periodic motion (such as the motion of an ocean wave). To obtain information from a model, we often need to solve equations. If the model involves trigonometric functions, we need to solve trigonometric equations. Solving trigonometric equations often involves using trigonometric identities. We've already encountered some basic trigonometric identities in the preceding chapters. We begin this chapter by finding many new identities.

7.1 Trigonometric Identities

| Simplifying Trigonometric Expressions ▶ Proving Trigonometric Identities

We begin by listing some of the basic trigonometric identities. We studied most of these in Chapters 5 and 6; you are asked to prove the cofunction identities in Exercise 102.

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \sec\left(\frac{\pi}{2} - u\right) = \csc u$$

$$\cos\left(\frac{\pi}{2} - u\right) = \sin u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

▼ Simplifying Trigonometric Expressions

Identities enable us to write the same expression in different ways. It is often possible to rewrite a complicated-looking expression as a much simpler one. To simplify algebraic expressions, we used factoring, common denominators, and the Special Product Formulas. To simplify trigonometric expressions, we use these same techniques together with the fundamental trigonometric identities.

EXAMPLE 1 | Simplifying a Trigonometric Expression

Simplify the expression $\cos t + \tan t \sin t$.

SOLUTION We start by rewriting the expression in terms of sine and cosine:

$$\begin{aligned}
\cos t + \tan t \sin t &= \cos t + \left(\frac{\sin t}{\cos t}\right)\sin t && \text{Reciprocal identity} \\
&= \frac{\cos^2 t + \sin^2 t}{\cos t} && \text{Common denominator} \\
&= \frac{1}{\cos t} && \text{Pythagorean identity} \\
&= \sec t && \text{Reciprocal identity}
\end{aligned}$$

NOW TRY EXERCISE 3 ■

EXAMPLE 2 | Simplifying by Combining Fractions

Simplify the expression $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1 + \sin\theta}$.

SOLUTION We combine the fractions by using a common denominator:

$$\begin{aligned}
\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} &= \frac{\sin\theta\,(1+\sin\theta) + \cos^2\theta}{\cos\theta\,(1+\sin\theta)} && \text{Common denominator}\\
&= \frac{\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta\,(1+\sin\theta)} && \text{Distribute } \sin\theta\\
&= \frac{\sin\theta + 1}{\cos\theta\,(1+\sin\theta)} && \text{Pythagorean identity}\\
&= \frac{1}{\cos\theta} = \sec\theta && \text{Cancel and use reciprocal identity}
\end{aligned}$$

NOW TRY EXERCISE **21**

Proving Trigonometric Identities

Many identities follow from the fundamental identities. In the examples that follow, we learn how to prove that a given trigonometric equation is an identity, and in the process we will see how to discover new identities.

First, it's easy to decide when a given equation is *not* an identity. All we need to do is show that the equation does not hold for some value of the variable (or variables). Thus the equation

$$\sin x + \cos x = 1$$

is not an identity, because when $x = \pi/4$, we have

$$\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \neq 1$$

To verify that a trigonometric equation is an identity, we transform one side of the equation into the other side by a series of steps, each of which is itself an identity.

GUIDELINES FOR PROVING TRIGONOMETRIC IDENTITIES

1. **Start with one side.** Pick one side of the equation and write it down. Your goal is to transform it into the other side. It's usually easier to start with the more complicated side.
2. **Use known identities.** Use algebra and the identities you know to change the side you started with. Bring fractional expressions to a common denominator, factor, and use the fundamental identities to simplify expressions.
3. **Convert to sines and cosines.** If you are stuck, you may find it helpful to rewrite all functions in terms of sines and cosines.

Warning: To prove an identity, we do *not* just perform the same operations on both sides of the equation. For example, if we start with an equation that is not an identity, such as

$$\sin x = -\sin x \tag{1}$$

and square both sides, we get the equation

$$\sin^2 x = \sin^2 x \tag{2}$$

which is clearly an identity. Does this mean that the original equation is an identity? Of course not. The problem here is that the operation of squaring is not **reversible** in the sense that we cannot arrive back at (1) from (2) by taking square roots (reversing the procedure). Only operations that are reversible will necessarily transform an identity into an identity.

EXAMPLE 3 | Proving an Identity by Rewriting in Terms of Sine and Cosine

Consider the equation $\cos\theta\,(\sec\theta - \cos\theta) = \sin^2\theta$.

(a) Verify algebraically that the equation is an identity.

(b) Confirm graphically that the equation is an identity.

SOLUTION

(a) The left-hand side looks more complicated, so we start with it and try to transform it into the right-hand side:

$$\begin{aligned}
\text{LHS} &= \cos\theta\,(\sec\theta - \cos\theta) \\
&= \cos\theta\left(\frac{1}{\cos\theta} - \cos\theta\right) && \text{Reciprocal identity} \\
&= 1 - \cos^2\theta && \text{Expand} \\
&= \sin^2\theta = \text{RHS} && \text{Pythagorean identity}
\end{aligned}$$

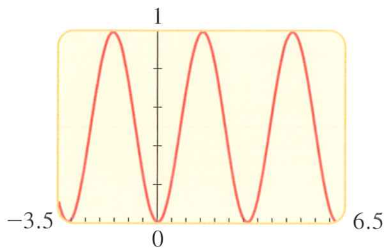


FIGURE 1

(b) We graph each side of the equation to see whether the graphs coincide. From Figure 1 we see that the graphs of $y = \cos\theta\,(\sec\theta - \cos\theta)$ and $y = \sin^2\theta$ are identical. This confirms that the equation is an identity.

NOW TRY EXERCISE **27**

In Example 3 it isn't easy to see how to change the right-hand side into the left-hand side, but it's definitely possible. Simply notice that each step is reversible. In other words, if we start with the last expression in the proof and work backward through the steps, the right-hand side is transformed into the left-hand side. You will probably agree, however, that it's more difficult to prove the identity this way. That's why it's often better to change the more complicated side of the identity into the simpler side.

EXAMPLE 4 | Proving an Identity by Combining Fractions

Verify the identity

$$2\tan x\sec x = \frac{1}{1-\sin x} - \frac{1}{1+\sin x}$$

SOLUTION Finding a common denominator and combining the fractions on the right-hand side of this equation, we get

$$\begin{aligned}
\text{RHS} &= \frac{1}{1-\sin x} - \frac{1}{1+\sin x} \\
&= \frac{(1+\sin x) - (1-\sin x)}{(1-\sin x)(1+\sin x)} && \text{Common denominator} \\
&= \frac{2\sin x}{1-\sin^2 x} && \text{Simplify} \\
&= \frac{2\sin x}{\cos^2 x} && \text{Pythagorean identity} \\
&= 2\,\frac{\sin x}{\cos x}\left(\frac{1}{\cos x}\right) && \text{Factor} \\
&= 2\tan x\sec x = \text{LHS} && \text{Reciprocal identities}
\end{aligned}$$

NOW TRY EXERCISE **79**

See the Prologue: *Principles of Problem Solving*, pages P1–P4.

In Example 5 we introduce "something extra" to the problem by multiplying the numerator and the denominator by a trigonometric expression, chosen so that we can simplify the result.

EXAMPLE 5 | Proving an Identity by Introducing Something Extra

Verify the identity $\dfrac{\cos u}{1 - \sin u} = \sec u + \tan u$.

SOLUTION We start with the left-hand side and multiply the numerator and denominator by $1 + \sin u$:

We multiply by $1 + \sin u$ because we know by the difference of squares formula that $(1 - \sin u)(1 + \sin u) = 1 - \sin^2 u$, and this is just $\cos^2 u$, a simpler expression.

$$\begin{aligned}
\text{LHS} &= \frac{\cos u}{1 - \sin u} \\
&= \frac{\cos u}{1 - \sin u} \cdot \frac{1 + \sin u}{1 + \sin u} && \text{Multiply numerator and denominator by } 1 + \sin u \\
&= \frac{\cos u\,(1 + \sin u)}{1 - \sin^2 u} && \text{Expand denominator} \\
&= \frac{\cos u\,(1 + \sin u)}{\cos^2 u} && \text{Pythagorean identity} \\
&= \frac{1 + \sin u}{\cos u} && \text{Cancel common factor} \\
&= \frac{1}{\cos u} + \frac{\sin u}{\cos u} && \text{Separate into two fractions} \\
&= \sec u + \tan u && \text{Reciprocal identities}
\end{aligned}$$

NOW TRY EXERCISE **53**

Here is another method for proving that an equation is an identity. If we can transform each side of the equation *separately*, by way of identities, to arrive at the same result, then the equation is an identity. Example 6 illustrates this procedure.

EXAMPLE 6 | Proving an Identity by Working with Both Sides Separately

Verify the identity $\dfrac{1 + \cos\theta}{\cos\theta} = \dfrac{\tan^2\theta}{\sec\theta - 1}$.

SOLUTION We prove the identity by changing each side separately into the same expression. Supply the reasons for each step:

$$\text{LHS} = \frac{1 + \cos\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\cos\theta}{\cos\theta} = \sec\theta + 1$$

$$\text{RHS} = \frac{\tan^2\theta}{\sec\theta - 1} = \frac{\sec^2\theta - 1}{\sec\theta - 1} = \frac{(\sec\theta - 1)(\sec\theta + 1)}{\sec\theta - 1} = \sec\theta + 1$$

It follows that LHS = RHS, so the equation is an identity.

NOW TRY EXERCISE **81**

EUCLID (circa 300 B.C.) taught in Alexandria. His *Elements* is the most widely influential scientific book in history. For 2000 years it was the standard introduction to geometry in the schools, and for many generations it was considered the best way to develop logical reasoning. Abraham Lincoln, for instance, studied the *Elements* as a way to sharpen his mind. The story is told that King Ptolemy once asked Euclid if there was a faster way to learn geometry than through the *Elements*. Euclid replied that there is "no royal road to geometry"—meaning by this that mathematics does not respect wealth or social status. Euclid was revered in his own time and was referred to by the title "The Geometer" or "The Writer of the *Elements*." The greatness of the *Elements* stems from its precise, logical, and systematic treatment of geometry. For dealing with equality, Euclid lists the following rules, which he calls "common notions."

1. Things that are equal to the same thing are equal to each other.
2. If equals are added to equals, the sums are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things that coincide with one another are equal.
5. The whole is greater than the part.

We conclude this section by describing the technique of *trigonometric substitution*, which we use to convert algebraic expressions to trigonometric ones. This is often useful in calculus, for instance, in finding the area of a circle or an ellipse.

EXAMPLE 7 | Trigonometric Substitution

Substitute $\sin\theta$ for x in the expression $\sqrt{1-x^2}$ and simplify. Assume that $0 \le \theta \le \pi/2$.

SOLUTION Setting $x = \sin\theta$, we have

$$\begin{aligned}\sqrt{1-x^2} &= \sqrt{1-\sin^2\theta} && \text{Substitute } x = \sin\theta\\ &= \sqrt{\cos^2\theta} && \text{Pythagorean identity}\\ &= \cos\theta && \text{Take square root}\end{aligned}$$

The last equality is true because $\cos\theta \ge 0$ for the values of θ in question.

NOW TRY EXERCISE **91**

7.1 EXERCISES

CONCEPTS

1. An equation is called an identity if it is valid for ________ values of the variable. The equation $2x = x + x$ is an algebraic identity, and the equation $\sin^2 x + \cos^2 x =$ ________ is a trigonometric identity.

2. For any x it is true that $\cos(-x)$ has the same value as $\cos x$. We express this fact as the identity ______________.

SKILLS

3–12 ■ Write the trigonometric expression in terms of sine and cosine, and then simplify.

3. $\cos t \tan t$
4. $\cos t \csc t$
5. $\sin\theta \sec\theta$
6. $\tan\theta \csc\theta$
7. $\tan^2 x - \sec^2 x$
8. $\dfrac{\sec x}{\csc x}$
9. $\sin u + \cot u \cos u$
10. $\cos^2\theta\,(1 + \tan^2\theta)$
11. $\dfrac{\sec\theta - \cos\theta}{\sin\theta}$
12. $\dfrac{\cot\theta}{\csc\theta - \sin\theta}$

13–26 ■ Simplify the trigonometric expression.

13. $\dfrac{\sin x \sec x}{\tan x}$
14. $\cos^3 x + \sin^2 x \cos x$
15. $\dfrac{1 + \cos y}{1 + \sec y}$
16. $\dfrac{\tan x}{\sec(-x)}$
17. $\dfrac{\sec^2 x - 1}{\sec^2 x}$
18. $\dfrac{\sec x - \cos x}{\tan x}$
19. $\dfrac{1 + \csc x}{\cos x + \cot x}$
20. $\dfrac{\sin x}{\csc x} + \dfrac{\cos x}{\sec x}$
21. $\dfrac{1 + \sin u}{\cos u} + \dfrac{\cos u}{1 + \sin u}$
22. $\tan x \cos x \csc x$
23. $\dfrac{2 + \tan^2 x}{\sec^2 x} - 1$
24. $\dfrac{1 + \cot A}{\csc A}$
25. $\tan\theta + \cos(-\theta) + \tan(-\theta)$
26. $\dfrac{\cos x}{\sec x + \tan x}$

27–28 ■ Consider the given equation. **(a)** Verify algebraically that the equation is an identity. **(b)** Confirm graphically that the equation is an identity.

27. $\dfrac{\cos x}{\sec x \sin x} = \csc x - \sin x$
28. $\dfrac{\tan y}{\csc y} = \sec y - \cos y$

29–90 ■ Verify the identity.

29. $\dfrac{\sin\theta}{\tan\theta} = \cos\theta$
30. $\dfrac{\tan x}{\sec x} = \sin x$
31. $\dfrac{\cos u \sec u}{\tan u} = \cot u$
32. $\dfrac{\cot x \sec x}{\csc x} = 1$
33. $\sin B + \cos B \cot B = \csc B$
34. $\cos(-x) - \sin(-x) = \cos x + \sin x$
35. $\cot(-\alpha)\cos(-\alpha) + \sin(-\alpha) = -\csc\alpha$
36. $\csc x\,[\csc x + \sin(-x)] = \cot^2 x$
37. $\tan\theta + \cot\theta = \sec\theta\csc\theta$
38. $(\sin x + \cos x)^2 = 1 + 2\sin x\cos x$
39. $(1 - \cos\beta)(1 + \cos\beta) = \dfrac{1}{\csc^2\beta}$
40. $\dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\csc x} = 1$
41. $\dfrac{(\sin x + \cos x)^2}{\sin^2 x - \cos^2 x} = \dfrac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2}$
42. $(\sin x + \cos x)^4 = (1 + 2\sin x\cos x)^2$
43. $\dfrac{\sec t - \cos t}{\sec t} = \sin^2 t$
44. $\dfrac{1 - \sin x}{1 + \sin x} = (\sec x - \tan x)^2$
45. $\dfrac{1}{1 - \sin^2 y} = 1 + \tan^2 y$
46. $\csc x - \sin x = \cos x \cot x$
47. $(\cot x - \csc x)(\cos x + 1) = -\sin x$
48. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$
49. $(1 - \cos^2 x)(1 + \cot^2 x) = 1$

50. $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$

51. $2\cos^2 x - 1 = 1 - 2\sin^2 x$

52. $(\tan y + \cot y)\sin y \cos y = 1$

53. $\dfrac{1 - \cos\alpha}{\sin\alpha} = \dfrac{\sin\alpha}{1 + \cos\alpha}$

54. $\sin^2\alpha + \cos^2\alpha + \tan^2\alpha = \sec^2\alpha$

55. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$

56. $\cot^2\theta\cos^2\theta = \cot^2\theta - \cos^2\theta$

57. $\dfrac{\sin x - 1}{\sin x + 1} = \dfrac{-\cos^2 x}{(\sin x + 1)^2}$

58. $\dfrac{\sin w}{\sin w + \cos w} = \dfrac{\tan w}{1 + \tan w}$

59. $\dfrac{(\sin t + \cos t)^2}{\sin t\cos t} = 2 + \sec t\csc t$

60. $\sec t\csc t(\tan t + \cot t) = \sec^2 t + \csc^2 t$

61. $\dfrac{1 + \tan^2 u}{1 - \tan^2 u} = \dfrac{1}{\cos^2 u - \sin^2 u}$

62. $\dfrac{1 + \sec^2 x}{1 + \tan^2 x} = 1 + \cos^2 x$

63. $\dfrac{\sec x}{\sec x - \tan x} = \sec x(\sec x + \tan x)$

64. $\dfrac{\sec x + \csc x}{\tan x + \cot x} = \sin x + \cos x$

65. $\sec v - \tan v = \dfrac{1}{\sec v + \tan v}$

66. $\dfrac{\sin A}{1 - \cos A} - \cot A = \csc A$

67. $\dfrac{\sin x + \cos x}{\sec x + \csc x} = \sin x\cos x$

68. $\dfrac{1 - \cos x}{\sin x} + \dfrac{\sin x}{1 - \cos x} = 2\csc x$

69. $\dfrac{\csc x - \cot x}{\sec x - 1} = \cot x$

70. $\dfrac{\csc^2 x - \cot^2 x}{\sec^2 x} = \cos^2 x$

71. $\tan^2 u - \sin^2 u = \tan^2 u\sin^2 u$

72. $\dfrac{\tan v\sin v}{\tan v + \sin v} = \dfrac{\tan v - \sin v}{\tan v\sin v}$

73. $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

74. $\dfrac{\cos\theta}{1 - \sin\theta} = \sec\theta + \tan\theta$

75. $\dfrac{\cos\theta}{1 - \sin\theta} = \dfrac{\sin\theta - \csc\theta}{\cos\theta - \cot\theta}$

76. $\dfrac{1 + \tan x}{1 - \tan x} = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

77. $\dfrac{\cos^2 t + \tan^2 t - 1}{\sin^2 t} = \tan^2 t$

78. $\dfrac{1}{1 - \sin x} - \dfrac{1}{1 + \sin x} = 2\sec x\tan x$

79. $\dfrac{1}{\sec x + \tan x} + \dfrac{1}{\sec x - \tan x} = 2\sec x$

80. $\dfrac{1 + \sin x}{1 - \sin x} - \dfrac{1 - \sin x}{1 + \sin x} = 4\tan x\sec x$

81. $(\tan x + \cot x)^2 = \sec^2 x + \csc^2 x$

82. $\tan^2 x - \cot^2 x = \sec^2 x - \csc^2 x$

83. $\dfrac{\sec u - 1}{\sec u + 1} = \dfrac{1 - \cos u}{1 + \cos u}$

84. $\dfrac{\cot x + 1}{\cot x - 1} = \dfrac{1 + \tan x}{1 - \tan x}$

85. $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x\cos x$

86. $\dfrac{\tan v - \cot v}{\tan^2 v - \cot^2 v} = \sin v\cos v$

87. $\dfrac{1 + \sin x}{1 - \sin x} = (\tan x + \sec x)^2$

88. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \tan x\tan y$

89. $(\tan x + \cot x)^4 = \csc^4 x\sec^4 x$

90. $(\sin\alpha - \tan\alpha)(\cos\alpha - \cot\alpha) = (\cos\alpha - 1)(\sin\alpha - 1)$

91–96 ■ Make the indicated trigonometric substitution in the given algebraic expression and simplify (see Example 7). Assume that $0 \le \theta < \pi/2$.

91. $\dfrac{x}{\sqrt{1 - x^2}}$, $x = \sin\theta$

92. $\sqrt{1 + x^2}$, $x = \tan\theta$

93. $\sqrt{x^2 - 1}$, $x = \sec\theta$

94. $\dfrac{1}{x^2\sqrt{4 + x^2}}$, $x = 2\tan\theta$

95. $\sqrt{9 - x^2}$, $x = 3\sin\theta$

96. $\dfrac{\sqrt{x^2 - 25}}{x}$, $x = 5\sec\theta$

97–100 ■ Graph f and g in the same viewing rectangle. Do the graphs suggest that the equation $f(x) = g(x)$ is an identity? Prove your answer.

97. $f(x) = \cos^2 x - \sin^2 x$, $g(x) = 1 - 2\sin^2 x$

98. $f(x) = \tan x(1 + \sin x)$, $g(x) = \dfrac{\sin x\cos x}{1 + \sin x}$

99. $f(x) = (\sin x + \cos x)^2$, $g(x) = 1$

100. $f(x) = \cos^4 x - \sin^4 x$, $g(x) = 2\cos^2 x - 1$

101. Show that the equation is not an identity.

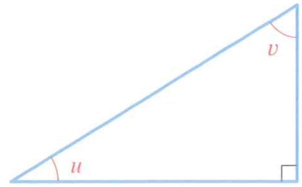
(a) $\sin 2x = 2\sin x$ **(b)** $\sin(x + y) = \sin x + \sin y$

(c) $\sec^2 x + \csc^2 x = 1$

(d) $\dfrac{1}{\sin x + \cos x} = \csc x + \sec x$

DISCOVERY ■ DISCUSSION ■ WRITING

102. **Cofunction Identities** In the right triangle shown, explain why $v = (\pi/2) - u$. Explain how you can obtain all six cofunction identities from this triangle for $0 < u < \pi/2$.



103. Graphs and Identities Suppose you graph two functions, f and g, on a graphing device and their graphs appear identical in the viewing rectangle. Does this prove that the equation $f(x) = g(x)$ is an identity? Explain.

104. Making Up Your Own Identity If you start with a trigonometric expression and rewrite it or simplify it, then setting the original expression equal to the rewritten expression yields a trigonometric identity. For instance, from Example 1 we get the identity

$$\cos t + \tan t \sin t = \sec t$$

Use this technique to make up your own identity, then give it to a classmate to verify.

7.2 Addition and Subtraction Formulas

Addition and Subtraction Formulas ▶ Evaluating Expressions Involving Inverse Trigonometric Functions ▶ Expressions of the form $A \sin x + B \cos x$

▼ Addition and Subtraction Formulas

We now derive identities for trigonometric functions of sums and differences.

ADDITION AND SUBTRACTION FORMULAS

Formulas for sine:
$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$
$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Formulas for cosine:
$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$
$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Formulas for tangent:
$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$
$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

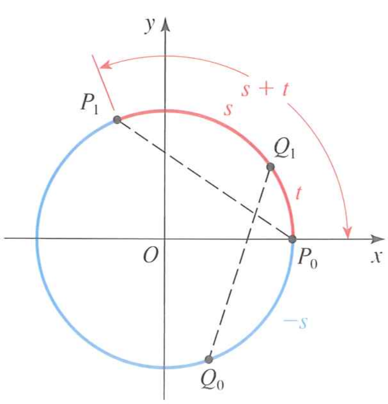


FIGURE 1

PROOF OF ADDITION FORMULA FOR COSINE To prove the formula $\cos(s + t) = \cos s \cos t - \sin s \sin t$, we use Figure 1. In the figure, the distances t, $s + t$, and $-s$ have been marked on the unit circle, starting at $P_0(1, 0)$ and terminating at Q_1, P_1, and Q_0, respectively. The coordinates of these points are

$$P_0(1, 0) \qquad Q_0(\cos(-s), \sin(-s))$$
$$P_1(\cos(s + t), \sin(s + t)) \qquad Q_1(\cos t, \sin t)$$

Since $\cos(-s) = \cos s$ and $\sin(-s) = -\sin s$, it follows that the point Q_0 has the coordinates $Q_0(\cos s, -\sin s)$. Notice that the distances between P_0 and P_1 and between Q_0 and Q_1 measured along the arc of the circle are equal. Since equal arcs are subtended by equal chords, it follows that $d(P_0, P_1) = d(Q_0, Q_1)$. Using the Distance Formula, we get

$$\sqrt{[\cos(s + t) - 1]^2 + [\sin(s + t) - 0]^2} = \sqrt{(\cos t - \cos s)^2 + (\sin t + \sin s)^2}$$

Squaring both sides and expanding, we have

These add to 1

$$\cos^2(s + t) - 2\cos(s + t) + 1 + \sin^2(s + t)$$
$$= \cos^2 t - 2\cos s \cos t + \cos^2 s + \sin^2 t + 2\sin s \sin t + \sin^2 s$$

These add to 1

These add to 1