DISCOVERY ■ DISCUSSION ■ WRITING

72. **Using a Calculator** To solve a certain problem, you need to find the sine of 4 rad. Your study partner uses his calculator and tells you that

$$\sin 4 = 0.0697564737$$

On your calculator you get

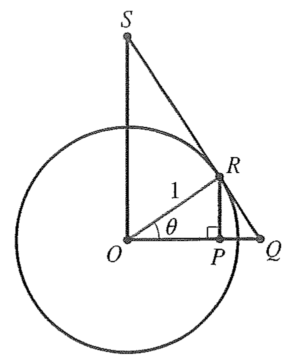
$$\sin 4 = -0.7568024953$$

What is wrong? What mistake did your partner make?

73. **Viète's Trigonometric Diagram** In the 16th century the French mathematician François Viète (see page 49) published the following remarkable diagram. Each of the six trigonometric functions of θ is equal to the length of a line segment in the figure. For instance, $\sin \theta = |PR|$, since from $\triangle OPR$ we see that

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{|PR|}{|OR|} = \frac{|PR|}{1} = |PR|$$

For each of the five other trigonometric functions, find a line segment in the figure whose length equals the value of the function at θ. (*Note:* The radius of the circle is 1, the center is O, segment QS is tangent to the circle at R, and $\angle SOQ$ is a right angle.)



DISCOVERY PROJECT **Similarity**

In this project we explore the idea of similarity and some of its consequences for any type of figure. You can find the project at the book companion website: **www.stewartmath.com**

6.4 Inverse Trigonometric Functions and Right Triangles

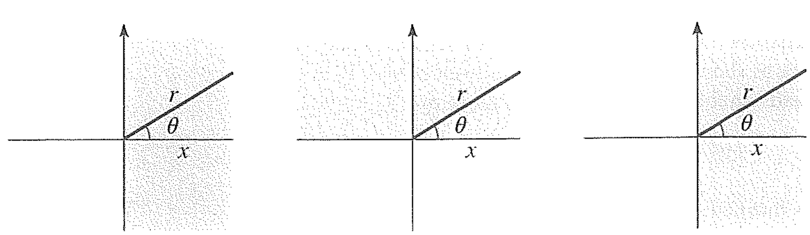
The Inverse Sine, Inverse Cosine, and Inverse Tangent Functions ▶ Solving for Angles in Right Triangles ▶ Evaluating Expressions Involving Inverse Trigonometric Functions

The graphs of the inverse trigonometric functions are studied in Section 5.5.

Recall that for a function to have an inverse, it must be one-to-one. Since the trigonometric functions are not one-to-one, they do not have inverses. So we restrict the domain of each of the trigonometric functions to intervals on which they attain all their values and on which they are one-to-one. The resulting functions have the same range as the original functions but are one-to-one.

▼ The Inverse Sine, Inverse Cosine, and Inverse Tangent Functions

Let's first consider the sine function. We restrict the domain of the sine function to angles θ with $-\pi/2 \le \theta \le \pi/2$. From Figure 1 we see that on this domain the sine function attains each of the values in the interval $[-1, 1]$ exactly once and so is one-to-one. Similarly, we restrict the domains of cosine and tangent as shown in Figure 1.



$\sin \theta = \frac{y}{r}$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\cos \theta = \frac{x}{r}$ $\quad 0 \le \theta \le \pi$

$\tan \theta = \frac{x}{y}$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

FIGURE 1 Restricted domains of the sine, cosine, and tangent functions

On these restricted domains we can define an inverse for each of these functions. By the definition of inverse function we have

$$\begin{aligned} \sin^{-1} x = y \quad &\Leftrightarrow \quad \sin y = x \\ \cos^{-1} x = y \quad &\Leftrightarrow \quad \cos y = x \\ \tan^{-1} x = y \quad &\Leftrightarrow \quad \tan y = x \end{aligned}$$

We summarize the domains and ranges of the inverse trigonometric functions in the following box.

THE INVERSE SINE, INVERSE COSINE, AND INVERSE TANGENT FUNCTIONS

The sine, cosine, and tangent functions on the restricted domains $[-\pi/2, \pi/2]$, $[0, \pi]$, and $(-\pi/2, \pi/2)$, respectively, are one-to one and so have inverses. The inverse functions have domain and range as follows.

Function	Domain	Range
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}$	$\mathbb{R}$	$(-\pi/2, \pi/2)$

The functions $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ are sometimes called **arcsine, arccosine,** and **arctangent**, respectively.

Since these are inverse functions, they reverse the rule of the original function. For example, since $\sin \pi/6 = \frac{1}{2}$, it follows that $\sin^{-1} \frac{1}{2} = \pi/6$. The following example gives further illustrations.

EXAMPLE 1 | Evaluating Inverse Trigonometric Functions

Find the exact value.

(a) $\sin^{-1} \dfrac{\sqrt{3}}{2}$ (b) $\cos^{-1}(-\frac{1}{2})$ (c) $\tan^{-1} 1$

SOLUTION

(a) The angle in the interval $[-\pi/2, \pi/2]$ whose sine is $\sqrt{3}/2$ is $\pi/3$. Thus $\sin^{-1}(\sqrt{3}/2) = \pi/3$.

(b) The angle in the interval $[0, \pi]$ whose cosine is $-\frac{1}{2}$ is $2\pi/3$. Thus $\cos^{-1}(-\frac{1}{2}) = 2\pi/3$.

(c) The angle in the interval $[-\pi/2, \pi/2]$ whose tangent is 1 is $\pi/4$. Thus $\tan^{-1} 1 = \pi/4$.

NOW TRY EXERCISE 3

EXAMPLE 2 | Evaluating Inverse Trigonometric Functions

Find approximate values for the given expression.

(a) $\sin^{-1}(0.71)$ (b) $\tan^{-1}(2)$ (c) $\cos^{-1} 2$

SOLUTION We use a calculator to approximate these values.

(a) Using the $\boxed{\text{INV}}\boxed{\text{SIN}}$, or $\boxed{\text{SIN}^{-1}}$, or $\boxed{\text{ARC}}\boxed{\text{SIN}}$ key(s) on the calculator (with the calculator in radian mode), we get

$$\sin^{-1}(0.71) \approx 0.78950$$

(b) Using the INV TAN, or TAN⁻¹, or ARC TAN key(s) on the calculator (with the calculator in radian mode), we get

$$\tan^{-1} 2 \approx 1.10715$$

(c) Since $2 > 1$, it is not in the domain of $\cos^{-1}$, so $\cos^{-1} 2$ is not defined.

NOW TRY EXERCISES **7, 11,** AND **13**

Solving for Angles in Right Triangles

In Section 6.2 we solved triangles by using the trigonometric functions to find the unknown sides. We now use the inverse trigonometric functions to solve for *angles* in a right triangle.

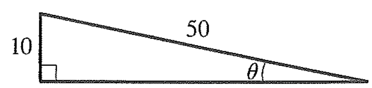


FIGURE 2

EXAMPLE 3 | Finding an Angle in a Right Triangle

Find the angle θ in the triangle shown in Figure 2.

SOLUTION Since θ is the angle opposite the side of length 10 and the hypotenuse has length 50, we have

$$\sin\theta = \frac{10}{50} = \frac{1}{5} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}}$$

Now we can use $\sin^{-1}$ to find θ:

$$\theta = \sin^{-1}\tfrac{1}{5} \qquad \text{Definition of } \sin^{-1}$$

$$\theta \approx 11.5^\circ \qquad \text{Calculator (in degree mode)}$$

NOW TRY EXERCISE **15**

EXAMPLE 4 | Solving for an Angle in a Right Triangle

A 40-ft ladder leans against a building. If the base of the ladder is 6 ft from the base of the building, what is the angle formed by the ladder and the building?

SOLUTION First we sketch a diagram as in Figure 3. If θ is the angle between the ladder and the building, then

$$\sin\theta = \frac{6}{40} = 0.15 \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}}$$

Now we use $\sin^{-1}$ to find θ:

$$\theta = \sin^{-1}(0.15) \qquad \text{Definition of } \sin^{-1}$$

$$\theta \approx 8.6^\circ \qquad \text{Calculator (in degree mode)}$$

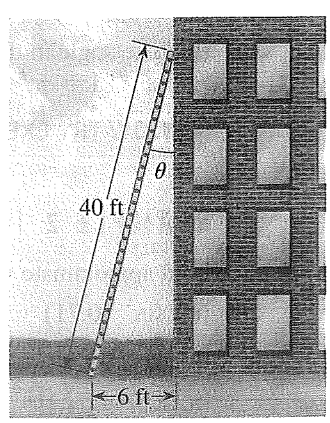


FIGURE 3

NOW TRY EXERCISE **37**

EXAMPLE 5 | The Angle of a Beam of Light

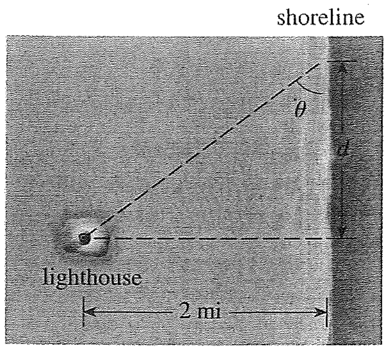


FIGURE 4

A lighthouse is located on an island that is 2 mi off a straight shoreline (see Figure 4). Express the angle formed by the beam of light and the shoreline in terms of the distance d in the figure.

SOLUTION From the figure we see that

$$\tan\theta = \frac{2}{d} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

Taking the inverse tangent of both sides, we get

$$\tan^{-1}(\tan\theta) = \tan^{-1}\left(\frac{2}{d}\right) \qquad \text{Take } \tan^{-1} \text{ of both sides}$$

$$\theta = \tan^{-1}\left(\frac{2}{d}\right) \qquad \text{Property of inverse functions: } \tan^{-1}(\tan\theta) = \theta$$

NOW TRY EXERCISE **39**

In Section 6.5 we learn how to solve any triangle (not necessarily a right triangle). The angles in a triangle are always in the interval $[0, \pi]$ (or between 0° and 180°). We'll see that to solve such triangles we need to find all angles in the interval $[0, \pi]$ that have a specified sine or cosine. We do this in the next example.

EXAMPLE 6 | Solving a Basic Trigonometric Equation on an Interval

Find all angles θ between 0° and 180° satisfying the given equation.

(a) $\sin\theta = 0.4$ **(b)** $\cos\theta = 0.4$

SOLUTION

(a) We use $\sin^{-1}$ to find one solution in the interval $[-\pi/2, \pi/2]$.

$$\sin\theta = 0.4 \qquad \text{Equation}$$

$$\theta = \sin^{-1}(0.4) \qquad \text{Take } \sin^{-1} \text{ of each side}$$

$$\theta \approx 23.6° \qquad \text{Calculator (in degree mode)}$$

Another solution with θ between 0° and 180° is obtained by taking the supplement of the angle: $180° - 23.6° = 156.4°$ (see Figure 5). So the solutions of the equation with θ between 0° and 180° are

$$\theta \approx 23.6° \quad \text{and} \quad \theta \approx 156.4°$$

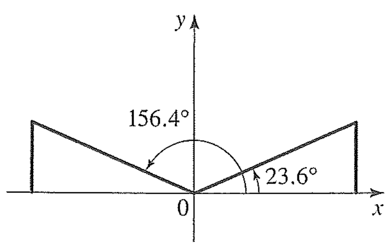


FIGURE 5

(b) The cosine function is one-to-one on the interval $[0, \pi]$, so there is only one solution of the equation with θ between 0° and 180°. We find that solution by taking $\cos^{-1}$ of each side.

$$\cos\theta = 0.4$$

$$\theta = \cos^{-1}(0.4) \qquad \text{Take } \cos^{-1} \text{ of each side}$$

$$\theta \approx 66.4° \qquad \text{Calculator (in degree mode)}$$

The solution is $\theta \approx 66.4°$

NOW TRY EXERCISES **23 AND 25**

Evaluating Expressions Involving Inverse Trigonometric Functions

Expressions like $\cos(\sin^{-1}x)$ arise in calculus. We find exact values of such expressions using trigonometric identities or right triangles.

EXAMPLE 7 | Composing Trigonometric Functions and Their Inverses

Find $\cos(\sin^{-1}\frac{3}{5})$.

SOLUTION 1

Let $\theta = \sin^{-1}\frac{3}{5}$. Then θ is the number in the interval $[-\pi/2, \pi/2]$ whose sine is $\frac{3}{5}$. Let's interpret θ as an angle and draw a right triangle with θ as one of its acute angles, with opposite side 3 and hypotenuse 5 (see Figure 6). The remaining leg of the triangle is found by the Pythagorean Theorem to be 4. From the figure we get

$$\begin{aligned} \cos(\sin^{-1}\tfrac{3}{5}) &= \cos\theta && \theta = \sin^{-1}\tfrac{3}{5} \\ &= \frac{4}{5} && \cos\theta = \frac{\text{adj}}{\text{hyp}} \end{aligned}$$

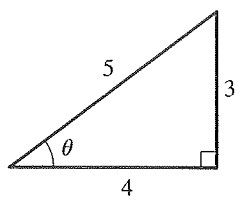


FIGURE 6 $\cos\theta = \dfrac{4}{5}$

So $\cos(\sin^{-1}\frac{3}{5}) = \frac{4}{5}$.

SOLUTION 2

It's easy to find $\sin(\sin^{-1}\frac{3}{5})$. In fact, by the cancellation properties of inverse functions, this value is exactly $\frac{3}{5}$. To find $\cos(\sin^{-1}\frac{3}{5})$, we first write the cosine function in terms of the sine function. Let $u = \sin^{-1}\frac{3}{5}$. Since $-\pi/2 \le u \le \pi/2$, $\cos u$ is positive, and we can write the following:

$$\begin{aligned} \cos u &= +\sqrt{1 - \sin^2 u} && \cos^2 u + \sin^2 u = 1 \\ &= \sqrt{1 - \sin^2(\sin^{-1}\tfrac{3}{5})} && u = \sin^{-1}\tfrac{3}{5} \\ &= \sqrt{1 - (\tfrac{3}{5})^2} && \text{Property of inverse functions: } \sin(\sin^{-1}\tfrac{3}{5}) = \tfrac{3}{5} \\ &= \sqrt{1 - \tfrac{9}{25}} = \sqrt{\tfrac{16}{25}} = \tfrac{4}{5} && \text{Calculate} \end{aligned}$$

So $\cos(\sin^{-1}\frac{3}{5}) = \frac{4}{5}$.

NOW TRY EXERCISE **27**

EXAMPLE 8 | Composing Trigonometric Functions and Their Inverses

Write $\sin(\cos^{-1}x)$ and $\tan(\cos^{-1}x)$ as algebraic expressions in x for $-1 \le x \le 1$.

SOLUTION 1

Let $\theta = \cos^{-1}x$; then $\cos\theta = x$. In Figure 7 we sketch a right triangle with an acute angle θ, adjacent side x, and hypotenuse 1. By the Pythagorean Theorem the remaining leg is $\sqrt{1 - x^2}$. From the figure we have

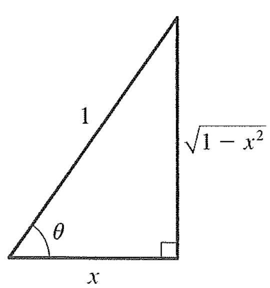


FIGURE 7 $\cos\theta = \dfrac{x}{1} = x$

$$\sin(\cos^{-1}x) = \sin\theta = \sqrt{1 - x^2} \qquad \text{and} \qquad \tan(\cos^{-1}x) = \tan\theta = \frac{\sqrt{1 - x^2}}{x}$$

SOLUTION 2

Let $u = \cos^{-1}x$. We need to find $\sin u$ and $\tan u$ in terms of x. As in Example 5 the idea here is to write sine and tangent in terms of cosine. Note that $0 \le u \le \pi$ because $u = \cos^{-1}x$. We have

$$\sin u = \pm\sqrt{1 - \cos^2 u} \qquad \text{and} \qquad \tan u = \frac{\sin u}{\cos u} = \frac{\pm\sqrt{1 - \cos^2 u}}{\cos u}$$

To choose the proper signs, note that u lies in the interval $[0, \pi]$ because $u = \cos^{-1} x$. Since $\sin u$ is positive on this interval, the $+$ sign is the correct choice. Substituting $u = \cos^{-1} x$ in the displayed equations and using the cancellation property $\cos(\cos^{-1} x) = x$, we get

$$\sin(\cos^{-1} x) = \sqrt{1 - x^2} \qquad \text{and} \qquad \tan(\cos^{-1} x) = \frac{\sqrt{1 - x^2}}{x}$$

NOW TRY EXERCISES **33** AND **35**

Note: In Solution 1 of Example 8 it might seem that because we are sketching a triangle, the angle $\theta = \cos^{-1} x$ must be acute. But it turns out that the triangle method works for any x. The domains and ranges of all six inverse trigonometric functions have been chosen in such a way that we can always use a triangle to find $S(T^{-1}(x))$, where S and T are any trigonometric functions.

6.4 EXERCISES

CONCEPTS

1. The inverse sine, inverse cosine, and inverse tangent functions have the followings domains and ranges.

(a) The function $\sin^{-1}$ has domain ________ and range ________.

(b) The function $\cos^{-1}$ has domain ________ and range ________.

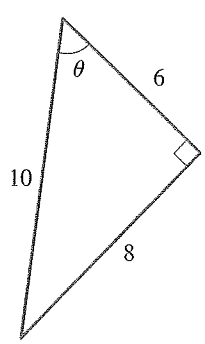
(c) The function $\tan^{-1}$ has domain ________ and range ________.

2. In the triangle shown, we can find the angle θ as follows:

(a) $\theta = \sin^{-1} \dfrac{\square}{\square}$

(b) $\theta = \cos^{-1} \dfrac{\square}{\square}$

(c) $\theta = \tan^{-1} \dfrac{\square}{\square}$



SKILLS

3–6 ■ Find the exact value of each expression, if it is defined.

3. (a) $\sin^{-1}\dfrac{1}{2}$ (b) $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ (c) $\tan^{-1}(-1)$

4. (a) $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ (b) $\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$ (c) $\tan^{-1}(-\sqrt{3})$

5. (a) $\sin^{-1}\left(-\dfrac{1}{2}\right)$ (b) $\cos^{-1}\dfrac{1}{2}$ (c) $\tan^{-1}\left(\dfrac{\sqrt{3}}{3}\right)$

6. (a) $\sin^{-1}(-1)$ (b) $\cos^{-1} 1$ (c) $\tan^{-1} 0$

7–14 ■ Use a calculator to find an approximate value of each expression rounded to five decimal places, if it is defined.

7. $\sin^{-1}(0.45)$ **8.** $\cos^{-1}(-0.75)$

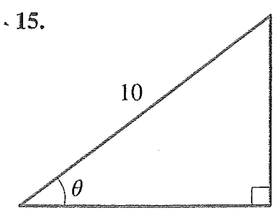
9. $\cos^{-1}(-\frac{1}{4})$ **10.** $\sin^{-1}\frac{1}{3}$

11. $\tan^{-1} 3$ **12.** $\tan^{-1}(-4)$

13. $\cos^{-1} 3$ **14.** $\sin^{-1}(-2)$

15–20 ■ Find the angle θ in degrees, rounded to one decimal.

15.



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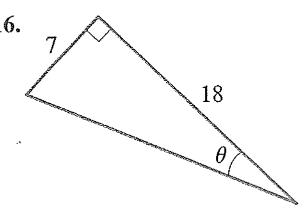

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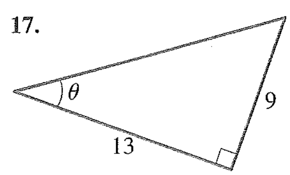

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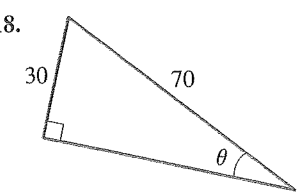

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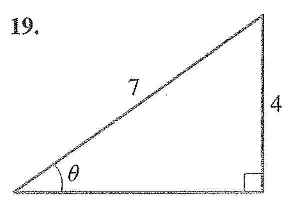

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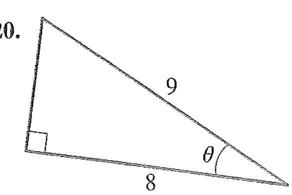


21–26 ■ Find all angles θ between 0° and 180° satisfying the given equation.

21. $\sin\theta = \dfrac{1}{2}$ **22.** $\sin\theta = \dfrac{\sqrt{3}}{2}$

23. $\sin\theta = 0.7$ 24. $\sin\theta = \frac{1}{4}$

25. $\cos\theta = 0.7$ 26. $\cos\theta = \frac{1}{9}$

27–32 ■ Find the exact value of the expression.

27. $\sin(\cos^{-1}\frac{3}{5})$ 28. $\tan(\sin^{-1}\frac{4}{5})$ 29. $\sec(\sin^{-1}\frac{12}{13})$

30. $\csc(\cos^{-1}\frac{7}{25})$ 31. $\tan(\sin^{-1}\frac{12}{13})$ 32. $\cot(\sin^{-1}\frac{2}{3})$

33–36 ■ Rewrite the expression as an algebraic expression in x.

33. $\cos(\sin^{-1}x)$ 34. $\sin(\tan^{-1}x)$

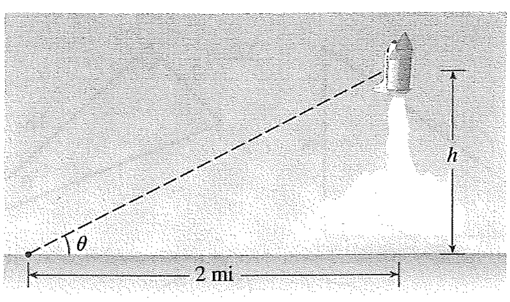
35. $\tan(\sin^{-1}x)$ 36. $\cos(\tan^{-1}x)$

APPLICATIONS

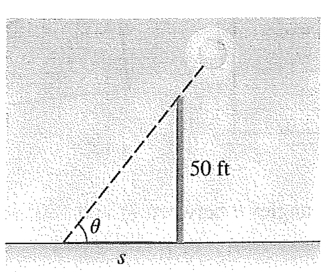
37. **Leaning Ladder** A 20-ft ladder is leaning against a building. If the base of the ladder is 6 ft from the base of the building, what is the angle of elevation of the ladder? How high does the ladder reach on the building?

38. **Angle of the Sun** A 96-ft tree casts a shadow that is 120 ft long. What is the angle of elevation of the sun?

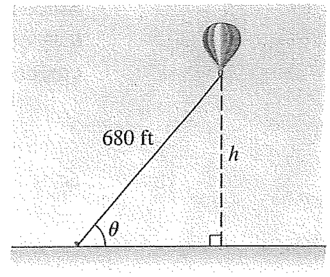
39. **Height of the Space Shuttle** An observer views the space shuttle from a distance of 2 mi from the launch pad.
   (a) Express the height of the space shuttle as a function of the angle of elevation θ.
   (b) Express the angle of elevation θ as a function of the height h of the space shuttle.



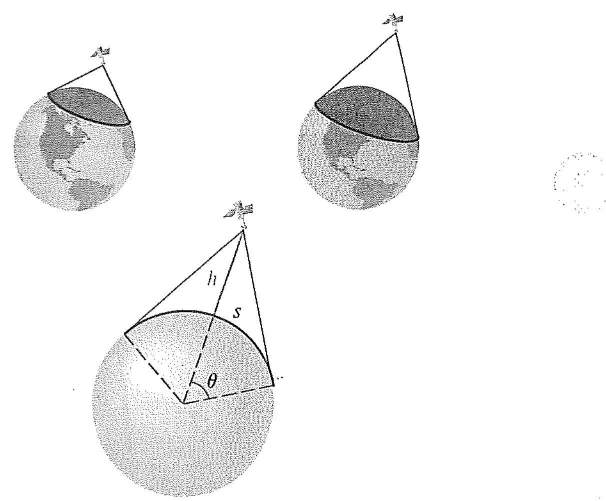
40. **Height of a Pole** A 50-ft pole casts a shadow as shown in the figure.
   (a) Express the angle of elevation θ of the sun as a function of the length s of the shadow.
   (b) Find the angle θ of elevation of the sun when the shadow is 20 ft long.



41. **Height of a Balloon** A 680-ft rope anchors a hot-air balloon as shown in the figure.
   (a) Express the angle θ as a function of the height h of the balloon.
   (b) Find the angle θ if the balloon is 500 ft high.



42. **View from a Satellite** The figures indicate that the higher the orbit of a satellite, the more of the earth the satellite can "see." Let θ, s, and h be as in the figure, and assume the earth is a sphere of radius 3960 mi.
   (a) Express the angle θ as a function of h.
   (b) Express the distance s as a function of θ.
   (c) Express the distance s as a function of h. [*Hint:* Find the composition of the functions in parts (a) and (b).]
   (d) If the satellite is 100 mi above the earth, what is the distance s that it can see?
   (e) How high does the satellite have to be to see both Los Angeles and New York, 2450 mi apart?



43. **Surfing the Perfect Wave** For a wave to be surfable, it can't break all at once. Robert Guza and Tony Bowen have shown that a wave has a surfable shoulder if it hits the shoreline at an angle θ given by

$$\theta = \sin^{-1}\left(\frac{1}{(2n+1)\tan\beta}\right)$$

where β is the angle at which the beach slopes down and where $n = 0, 1, 2, \ldots$.
   (a) For $\beta = 10°$, find θ when $n = 3$.
   (b) For $\beta = 15°$, find θ when $n = 2$, 3, and 4. Explain why the formula does not give a value for θ when $n = 0$ or 1.