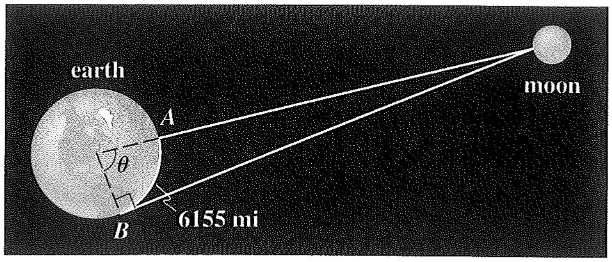
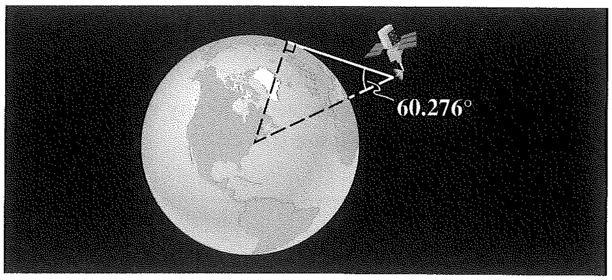
figure). Points A and B are 6155 mi apart, and the radius of the earth is 3960 mi.

(a) Find the angle θ in degrees.

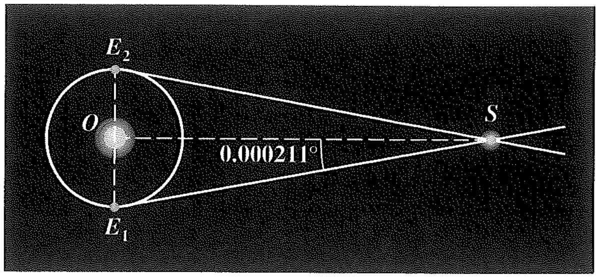
(b) Estimate the distance from point A to the moon.



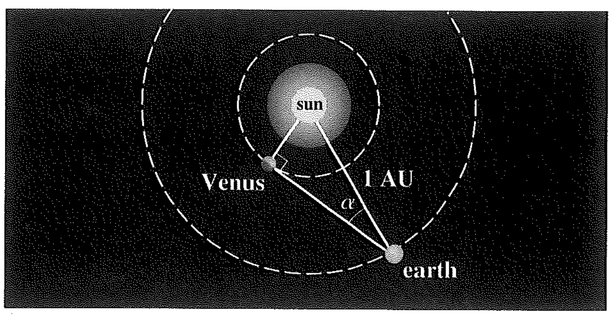
63. Radius of the Earth In Exercise 74 of Section 6.1 a method was given for finding the radius of the earth. Here is a more modern method: From a satellite 600 mi above the earth, it is observed that the angle formed by the vertical and the line of sight to the horizon is 60.276°. Use this information to find the radius of the earth.



64. Parallax To find the distance to nearby stars, the method of parallax is used. The idea is to find a triangle with the star at one vertex and with a base as large as possible. To do this, the star is observed at two different times exactly 6 months apart, and its apparent change in position is recorded. From these two observations, $\angle E_1SE_2$ can be calculated. (The times are chosen so that $\angle E_1SE_2$ is as large as possible, which guarantees that $\angle E_1OS$ is 90°.) The angle E_1SO is called the *parallax* of the star. Alpha Centauri, the star nearest the earth, has a parallax of 0.000211°. Estimate the distance to this star. (Take the distance from the earth to the sun to be 9.3×10^7 mi.)



65. Distance from Venus to the Sun The **elongation** α of a planet is the angle formed by the planet, earth, and sun (see the figure). When Venus achieves its maximum elongation of 46.3°, the earth, Venus, and the sun form a triangle with a right angle at Venus. Find the distance between Venus and the sun in astronomical units (AU). (By definition the distance between the earth and the sun is 1 AU.)



DISCOVERY ■ DISCUSSION ■ WRITING

66. Similar Triangles If two triangles are similar, what properties do they share? Explain how these properties make it possible to define the trigonometric ratios without regard to the size of the triangle.

6.3 Trigonometric Functions of Angles

Trigonometric Functions of Angles ▶ Evaluating Trigonometric Functions at Any Angle ▶ Trigonometric Identities ▶ Areas of Triangles

In the preceding section we defined the trigonometric ratios for acute angles. Here we extend the trigonometric ratios to all angles by defining the trigonometric functions of angles. With these functions we can solve practical problems that involve angles that are not necessarily acute.

Trigonometric Functions of Angles

Let POQ be a right triangle with acute angle θ as shown in Figure 1(a). Place θ in standard position as shown in Figure 1(b).

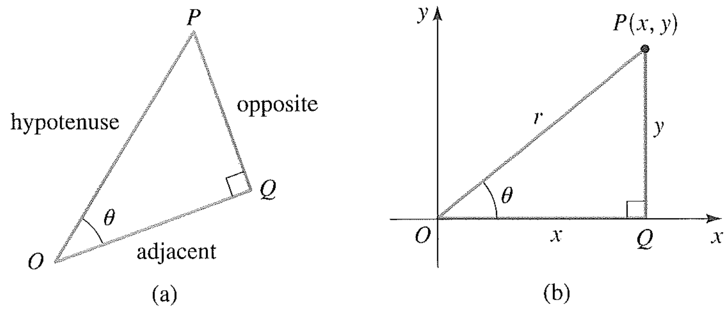


FIGURE 1

Then $P = P(x, y)$ is a point on the terminal side of θ. In triangle POQ, the opposite side has length y and the adjacent side has length x. Using the Pythagorean Theorem, we see that the hypotenuse has length $r = \sqrt{x^2 + y^2}$. So

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

The other trigonometric ratios can be found in the same way.

These observations allow us to extend the trigonometric ratios to any angle. We define the trigonometric functions of angles as follows (see Figure 2).

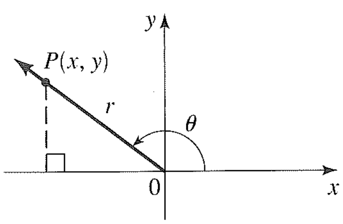


FIGURE 2

DEFINITION OF THE TRIGONOMETRIC FUNCTIONS

Let θ be an angle in standard position and let $P(x, y)$ be a point on the terminal side. If $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point $P(x, y)$, then

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

$$\csc\theta = \frac{r}{y} \quad (y \neq 0) \qquad \sec\theta = \frac{r}{x} \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0)$$

Since division by 0 is an undefined operation, certain trigonometric functions are not defined for certain angles. For example, $\tan 90° = y/x$ is undefined because $x = 0$. The angles for which the trigonometric functions may be undefined are the angles for which either the x- or y-coordinate of a point on the terminal side of the angle is 0. These are **quadrantal angles**—angles that are coterminal with the coordinate axes.

It is a crucial fact that the values of the trigonometric functions do *not* depend on the choice of the point $P(x, y)$. This is because if $P'(x', y')$ is any other point on the terminal side, as in Figure 3, then triangles POQ and $P'OQ'$ are similar.

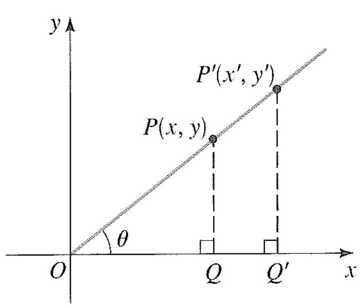


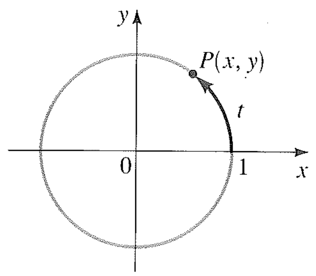
FIGURE 3

Evaluating Trigonometric Functions at Any Angle

From the definition we see that the values of the trigonometric functions are all positive if the angle θ has its terminal side in Quadrant I. This is because x and y are positive in this quadrant. [Of course, r is always positive, since it is simply the distance from the origin to the point $P(x, y)$.] If the terminal side of θ is in Quadrant II, however, then x is

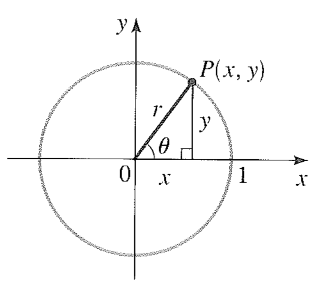
Relationship to the Trigonometric Functions of Real Numbers

You may have already studied the trigonometric functions defined using the unit circle (Chapter 5). To see how they relate to the trigonometric functions of an *angle*, let's start with the unit circle in the coordinate plane.



$P(x, y)$ is the terminal point determined by t.

Let $P(x, y)$ be the terminal point determined by an arc of length t on the unit circle. Then t subtends an angle θ at the center of the circle. If we drop a perpendicular from P onto the point Q on the x-axis, then triangle ΔOPQ is a right triangle with legs of length x and y, as shown in the figure.



Triangle OPQ is a right triangle.

Now, by the definition of the trigonometric functions of the *real number* t we have

$$\sin t = y$$

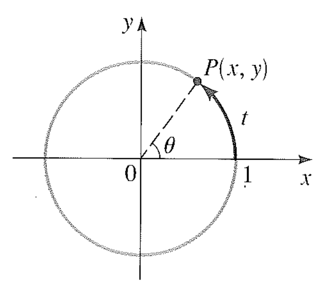
$$\cos t = x$$

By the definition of the trigonometric functions of the *angle* θ we have

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$

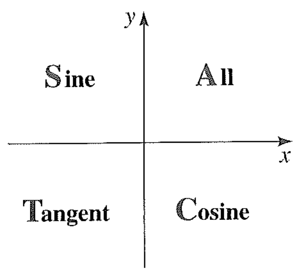
If θ is measured in radians, then $\theta = t$. (See the figure below.) Comparing the two ways of defining the trigonometric functions, we see that they are identical. In other words, as functions, they assign identical values to a given real number. (The real number is the radian measure of θ in one case or the length t of an arc in the other.)



The radian measure of angle θ is t.

Why then do we study trigonometry in two different ways? Because different applications require that we view the trigonometric functions differently. (See *Focus on Modeling*, pages 427, 489, and 533, and Sections 6.2, 6.5, and 6.6.)

The following mnemonic device can be used to remember which trigonometric functions are positive in each quadrant: All of them, Sine, Tangent, or Cosine.



You can remember this as "All Students Take Calculus."

negative and y is positive. Thus, in Quadrant II the functions $\sin\theta$ and $\csc\theta$ are positive, and all the other trigonometric functions have negative values. You can check the other entries in the following table.

SIGNS OF THE TRIGONOMETRIC FUNCTIONS

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

We now turn our attention to finding the values of the trigonometric functions for angles that are not acute.

EXAMPLE 1 | Finding Trigonometric Functions of Angles

Find **(a)** $\cos 135°$ and **(b)** $\tan 390°$.

SOLUTION

(a) From Figure 4 we see that $\cos 135° = -x/r$. But $\cos 45° = x/r$, and since $\cos 45° = \sqrt{2}/2$, we have

$$\cos 135° = -\frac{\sqrt{2}}{2}$$

(b) The angles 390° and 30° are coterminal. From Figure 5 it's clear that $\tan 390° = \tan 30°$ and, since $\tan 30° = \sqrt{3}/3$, we have

$$\tan 390° = \frac{\sqrt{3}}{3}$$

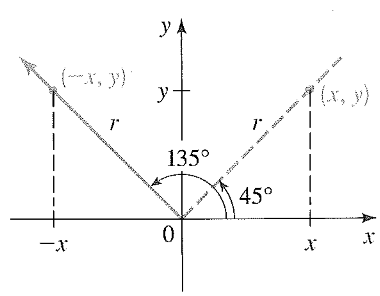


FIGURE 4

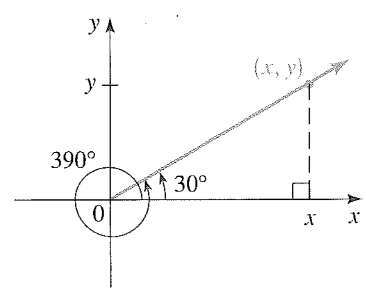


FIGURE 5

NOW TRY EXERCISES **11** AND **13**

From Example 1 we see that the trigonometric functions for angles that aren't acute have the same value, except possibly for sign, as the corresponding trigonometric functions of an acute angle. That acute angle will be called the *reference angle*.

REFERENCE ANGLE

Let θ be an angle in standard position. The **reference angle** $\bar{\theta}$ associated with θ is the acute angle formed by the terminal side of θ and the x-axis.

Figure 6 shows that to find a reference angle $\bar{\theta}$, it's useful to know the quadrant in which the terminal side of the angle θ lies.

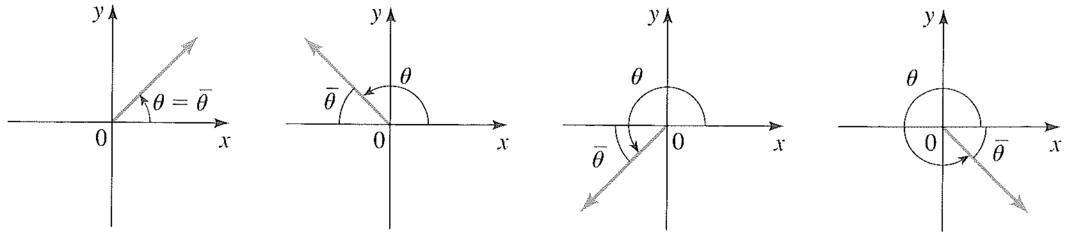


FIGURE 6 The reference angle $\bar{\theta}$ for an angle θ

EXAMPLE 2 | Finding Reference Angles

Find the reference angle for **(a)** $\theta = \dfrac{5\pi}{3}$ and **(b)** $\theta = 870°$.

SOLUTION

(a) The reference angle is the acute angle formed by the terminal side of the angle $5\pi/3$ and the x-axis (see Figure 7). Since the terminal side of this angle is in Quadrant IV, the reference angle is

$$\bar{\theta} = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$$

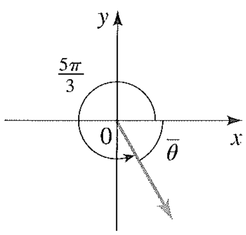


FIGURE 7

(b) The angles 870° and 150° are coterminal [because $870 - 2(360) = 150$]. Thus, the terminal side of this angle is in Quadrant II (see Figure 8). So the reference angle is

$$\bar{\theta} = 180° - 150° = 30°$$

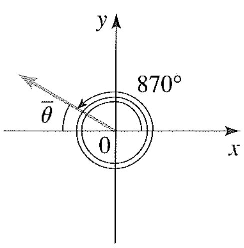


FIGURE 8

NOW TRY EXERCISES **3** AND **7**

EVALUATING TRIGONOMETRIC FUNCTIONS FOR ANY ANGLE

To find the values of the trigonometric functions for any angle θ, we carry out the following steps.

1. Find the reference angle $\bar{\theta}$ associated with the angle θ.
2. Determine the sign of the trigonometric function of θ by noting the quadrant in which θ lies.
3. The value of the trigonometric function of θ is the same, except possibly for sign, as the value of the trigonometric function of $\bar{\theta}$.

EXAMPLE 3 | Using the Reference Angle to Evaluate Trigonometric Functions

Find **(a)** sin 240° and **(b)** cot 495°.

SOLUTION

(a) This angle has its terminal side in Quadrant III, as shown in Figure 9. The reference angle is therefore $240° - 180° = 60°$, and the value of sin 240° is negative. Thus

$$\sin 240° = \underbrace{-}_{\text{Sign}}\underbrace{\sin 60°}_{\text{Reference angle}} = -\frac{\sqrt{3}}{2}$$

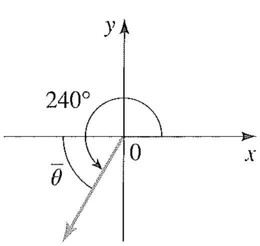


FIGURE 9

S | A
T | C

sin 240° is *negative*.

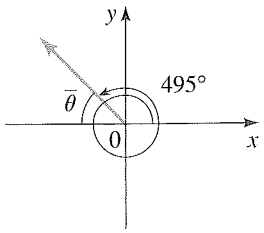


FIGURE 10

S | A
T | C

tan 495° is negative, so cot 495° is negative.

(b) The angle 495° is coterminal with the angle 135°, and the terminal side of this angle is in Quadrant II, as shown in Figure 10. So the reference angle is 180° − 135° = 45°, and the value of cot 495° is negative. We have

$$\cot 495° = \cot 135° = -\cot 45° = -1$$

Coterminal angles   Sign   Reference angle

NOW TRY EXERCISES **17** AND **19**

EXAMPLE 4 | Using the Reference Angle to Evaluate Trigonometric Functions

Find **(a)** $\sin \dfrac{16\pi}{3}$ and **(b)** $\sec\left(-\dfrac{\pi}{4}\right)$.

SOLUTION

(a) The angle $16\pi/3$ is coterminal with $4\pi/3$, and these angles are in Quadrant III (see Figure 11). Thus, the reference angle is $(4\pi/3) - \pi = \pi/3$. Since the value of sine is negative in Quadrant III, we have

$$\sin \frac{16\pi}{3} = \sin \frac{4\pi}{3} = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

Coterminal angles   Sign   Reference angle

(b) The angle $-\pi/4$ is in Quadrant IV, and its reference angle is $\pi/4$ (see Figure 12). Since secant is positive in this quadrant, we get

$$\sec\left(-\frac{\pi}{4}\right) = +\sec \frac{\pi}{4} = \sqrt{2}$$

Sign   Reference angle

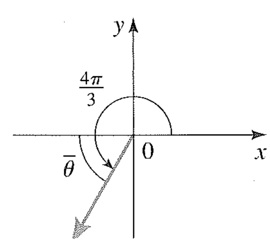


FIGURE 11

S | A
T | C

$\sin \frac{16\pi}{3}$ is negative.

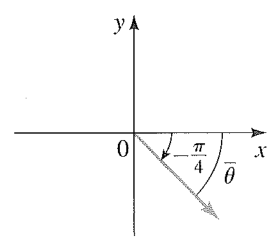


FIGURE 12

S | A
T | C

$\cos(-\frac{\pi}{4})$ is positive, so $\sec(-\frac{\pi}{4})$ is positive.

NOW TRY EXERCISES **23** AND **25**

Trigonometric Identities

The trigonometric functions of angles are related to each other through several important equations called **trigonometric identities**. We've already encountered the reciprocal identities. These identities continue to hold for any angle θ, provided that both sides of the equation are defined. The Pythagorean identities are a consequence of the Pythagorean Theorem.*

FUNDAMENTAL IDENTITIES

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

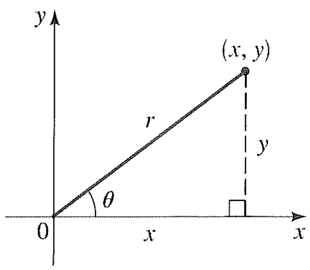


FIGURE 13

PROOF Let's prove the first Pythagorean identity. Using $x^2 + y^2 = r^2$ (the Pythagorean Theorem) in Figure 13, we have

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

Thus, $\sin^2\theta + \cos^2\theta = 1$. (Although the figure indicates an acute angle, you should check that the proof holds for all angles θ.) ■

See Exercises 61 and 62 for the proofs of the other two Pythagorean identities.

EXAMPLE 5 | Expressing One Trigonometric Function in Terms of Another

(a) Express $\sin\theta$ in terms of $\cos\theta$.

(b) Express $\tan\theta$ in terms of $\sin\theta$, where θ is in Quadrant II.

SOLUTION

(a) From the first Pythagorean identity we get

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

where the sign depends on the quadrant. If θ is in Quadrant I or II, then $\sin\theta$ is positive, and so

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

whereas if θ is in Quadrant III or IV, $\sin\theta$ is negative, and so

$$\sin\theta = -\sqrt{1 - \cos^2\theta}$$

(b) Since $\tan\theta = \sin\theta/\cos\theta$, we need to write $\cos\theta$ in terms of $\sin\theta$. By part (a)

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

and since $\cos\theta$ is negative in Quadrant II, the negative sign applies here. Thus

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{-\sqrt{1 - \sin^2\theta}}$$

NOW TRY EXERCISE **39** ■

---

*We follow the usual convention of writing $\sin^2\theta$ for $(\sin\theta)^2$. In general, we write $\sin^n\theta$ for $(\sin\theta)^n$ for all integers n except $n = -1$. The exponent $n = -1$ will be assigned another meaning in Section 6.4. Of course, the same convention applies to the other five trigonometric functions.

EXAMPLE 6 | Evaluating a Trigonometric Function

If $\tan\theta = \frac{2}{3}$ and θ is in Quadrant III, find $\cos\theta$.

SOLUTION 1 We need to write $\cos\theta$ in terms of $\tan\theta$. From the identity $\tan^2\theta + 1 = \sec^2\theta$ we get $\sec\theta = \pm\sqrt{\tan^2\theta + 1}$. In Quadrant III, $\sec\theta$ is negative, so

$$\sec\theta = -\sqrt{\tan^2\theta + 1}$$

If you wish to rationalize the denominator, you can express $\cos\theta$ as

$$-\frac{3}{\sqrt{13}}\cdot\frac{\sqrt{13}}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$$

Thus,

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{-\sqrt{\tan^2\theta + 1}}$$

$$= \frac{1}{-\sqrt{\left(\frac{2}{3}\right)^2 + 1}} = \frac{1}{-\sqrt{\frac{13}{9}}} = -\frac{3}{\sqrt{13}}$$

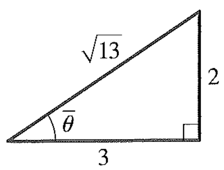


FIGURE 14

SOLUTION 2 This problem can be solved more easily by using the method of Example 2 of Section 6.2. Recall that, except for sign, the values of the trigonometric functions of any angle are the same as those of an acute angle (the reference angle). So, ignoring the sign for the moment, let's sketch a right triangle with an acute angle $\bar{\theta}$ satisfying $\tan\bar{\theta} = \frac{2}{3}$ (see Figure 14). By the Pythagorean Theorem the hypotenuse of this triangle has length $\sqrt{13}$. From the triangle in Figure 14 we immediately see that $\cos\bar{\theta} = 3/\sqrt{13}$. Since θ is in Quadrant III, $\cos\theta$ is negative, so

$$\cos\theta = -\frac{3}{\sqrt{13}}$$

NOW TRY EXERCISE **45**

EXAMPLE 7 | Evaluating Trigonometric Functions

If $\sec\theta = 2$ and θ is in Quadrant IV, find the other five trigonometric functions of θ.

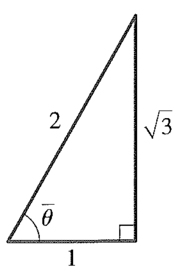


FIGURE 15

SOLUTION We sketch a triangle as in Figure 15 so that $\sec\bar{\theta} = 2$. Taking into account the fact that θ is in Quadrant IV, we get

$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad \cos\theta = \frac{1}{2} \qquad \tan\theta = -\sqrt{3}$$

$$\csc\theta = -\frac{2}{\sqrt{3}} \qquad \sec\theta = 2 \qquad \cot\theta = -\frac{1}{\sqrt{3}}$$

NOW TRY EXERCISE **47**

Areas of Triangles

We conclude this section with an application of the trigonometric functions that involves angles that are not necessarily acute. More extensive applications appear in the next two sections.

The area of a triangle is $\mathcal{A} = \frac{1}{2} \times \text{base} \times \text{height}$. If we know two sides and the included angle of a triangle, then we can find the height using the trigonometric functions, and from this we can find the area.

If θ is an acute angle, then the height of the triangle in Figure 16(a) is given by $h = b\sin\theta$. Thus the area is

$$\mathcal{A} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2}ab\sin\theta$$

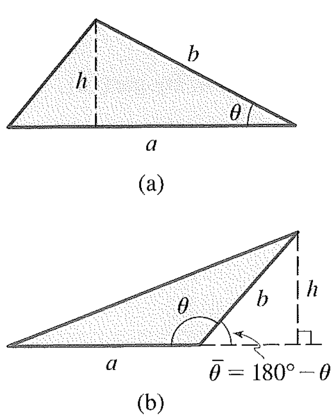


FIGURE 16

If the angle θ is not acute, then from Figure 16(b) we see that the height of the triangle is

$$h = b\sin(180° - \theta) = b\sin\theta$$

This is so because the reference angle of θ is the angle $180° - \theta$. Thus, in this case also, the area of the triangle is

$$\mathcal{A} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2}ab \sin\theta$$

AREA OF A TRIANGLE

The area $\mathcal{A}$ of a triangle with sides of lengths a and b and with included angle θ is

$$\mathcal{A} = \tfrac{1}{2}ab \sin\theta$$

EXAMPLE 8 | Finding the Area of a Triangle

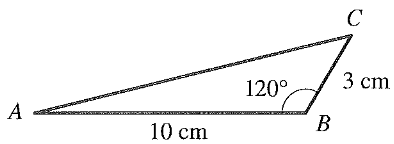


FIGURE 17

Find the area of triangle ABC shown in Figure 17.

SOLUTION The triangle has sides of length 10 cm and 3 cm, with included angle 120°. Therefore

$$\begin{aligned}\mathcal{A} &= \tfrac{1}{2}ab\sin\theta \\ &= \tfrac{1}{2}(10)(3)\sin 120° \\ &= 15\sin 60° && \text{Reference angle} \\ &= 15\frac{\sqrt{3}}{2} \approx 13 \text{ cm}^2\end{aligned}$$

NOW TRY EXERCISE **55**

6.3 EXERCISES

CONCEPTS

1. If the angle θ is in standard position and $P(x, y)$ is a point on the terminal side of θ, and r is the distance from the origin to P, then

$\sin\theta = \underline{\quad}$ $\cos\theta = \underline{\quad}$ $\tan\theta = \underline{\quad}$

2. The sign of a trigonometric function of θ depends on the ________ in which the terminal side of the angle θ lies.

In Quadrant II, $\sin\theta$ is ________ (positive / negative).

In Quadrant III, $\cos\theta$ is ________ (positive / negative).

In Quadrant IV, $\sin\theta$ is ________ (positive / negative).

SKILLS

3–10 ■ Find the reference angle for the given angle.

3. **(a)** 150° **(b)** 330° **(c)** $-30°$

4. **(a)** 120° **(b)** $-210°$ **(c)** 780°

5. **(a)** 225° **(b)** 810° **(c)** $-105°$

6. **(a)** 99° **(b)** $-199°$ **(c)** 359°

7. **(a)** $\dfrac{11\pi}{4}$ **(b)** $-\dfrac{11\pi}{6}$ **(c)** $\dfrac{11\pi}{3}$

8. **(a)** $\dfrac{4\pi}{3}$ **(b)** $\dfrac{33\pi}{4}$ **(c)** $-\dfrac{23\pi}{6}$

9. **(a)** $\dfrac{5\pi}{7}$ **(b)** -1.4π **(c)** 1.4

10. **(a)** 2.3π **(b)** 2.3 **(c)** -10π

11–34 ■ Find the exact value of the trigonometric function.

11. $\sin 150°$ **12.** $\sin 225°$ **13.** $\cos 210°$

14. $\cos(-60°)$ **15.** $\tan(-60°)$ **16.** $\sec 300°$

17. $\csc(-630°)$ **18.** $\cot 210°$ **19.** $\cos 570°$

20. $\sec 120°$ **21.** $\tan 750°$ **22.** $\cos 660°$

23. $\sin\dfrac{2\pi}{3}$ **24.** $\sin\dfrac{5\pi}{3}$ **25.** $\sin\dfrac{3\pi}{2}$

26. $\cos\dfrac{7\pi}{3}$ **27.** $\cos\left(-\dfrac{7\pi}{3}\right)$ **28.** $\tan\dfrac{5\pi}{6}$

29. $\sec\dfrac{17\pi}{3}$ **30.** $\csc\dfrac{5\pi}{4}$ **31.** $\cot\left(-\dfrac{\pi}{4}\right)$

32. $\cos\dfrac{7\pi}{4}$ **33.** $\tan\dfrac{5\pi}{2}$ **34.** $\sin\dfrac{11\pi}{6}$

35–38 ■ Find the quadrant in which θ lies from the information given.

35. $\sin\theta < 0$ and $\cos\theta < 0$

36. $\tan\theta < 0$ and $\sin\theta < 0$

37. $\sec\theta > 0$ and $\tan\theta < 0$

38. $\csc\theta > 0$ and $\cos\theta < 0$

39–44 ■ Write the first trigonometric function in terms of the second for θ in the given quadrant.

39. $\tan\theta$, $\cos\theta$; θ in Quadrant III

40. $\cot\theta$, $\sin\theta$; θ in Quadrant II

41. $\cos\theta$, $\sin\theta$; θ in Quadrant IV

42. $\sec\theta$, $\sin\theta$; θ in Quadrant I

43. $\sec\theta$, $\tan\theta$; θ in Quadrant II

44. $\csc\theta$, $\cot\theta$; θ in Quadrant III

45–52 ■ Find the values of the trigonometric functions of θ from the information given.

45. $\sin\theta = \frac{3}{5}$, θ in Quadrant II

46. $\cos\theta = -\frac{7}{12}$, θ in Quadrant III

47. $\tan\theta = -\frac{3}{4}$, $\cos\theta > 0$

48. $\sec\theta = 5$, $\sin\theta < 0$

49. $\csc\theta = 2$, θ in Quadrant I

50. $\cot\theta = \frac{1}{4}$, $\sin\theta < 0$

51. $\cos\theta = -\frac{2}{7}$, $\tan\theta < 0$

52. $\tan\theta = -4$, $\sin\theta > 0$

53. If $\theta = \pi/3$, find the value of each expression.
(a) $\sin 2\theta$, $2\sin\theta$ **(b)** $\sin\frac{1}{2}\theta$, $\frac{1}{2}\sin\theta$
(c) $\sin^2\theta$, $\sin(\theta^2)$

54. Find the area of a triangle with sides of length 7 and 9 and included angle 72°.

55. Find the area of a triangle with sides of length 10 and 22 and included angle 10°.

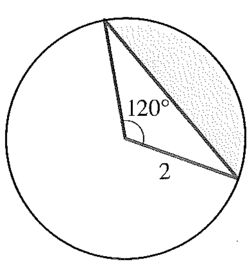
56. Find the area of an equilateral triangle with side of length 10.

57. A triangle has an area of 16 in^2, and two of the sides of the triangle have lengths 5 in. and 7 in. Find the angle included by these two sides.

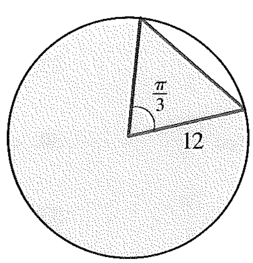
58. An isosceles triangle has an area of 24 cm^2, and the angle between the two equal sides is $5\pi/6$. What is the length of the two equal sides?

59–60 ■ Find the area of the shaded region in the figure.

59.



60.



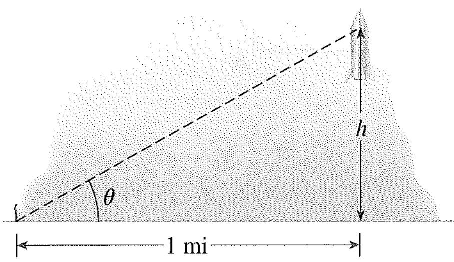
61. Use the first Pythagorean identity to prove the second. [*Hint:* Divide by $\cos^2\theta$.]

62. Use the first Pythagorean identity to prove the third.

APPLICATIONS

63. Height of a Rocket A rocket fired straight up is tracked by an observer on the ground a mile away.
(a) Show that when the angle of elevation is θ, the height of the rocket in feet is $h = 5280\tan\theta$.
(b) Complete the table to find the height of the rocket at the given angles of elevation.

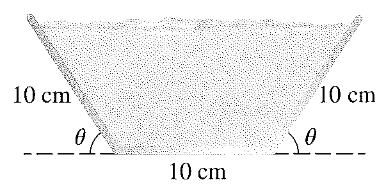
θ	20°	60°	80°	85°
h				



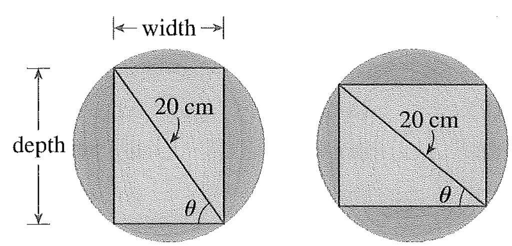
64. Rain Gutter A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ.
(a) Show that the cross-sectional area of the gutter is modeled by the function

$$A(\theta) = 100\sin\theta + 100\sin\theta\cos\theta$$

(b) Graph the function A for $0 \le \theta \le \pi/2$.
(c) For what angle θ is the largest cross-sectional area achieved?



65. Wooden Beam A rectangular beam is to be cut from a cylindrical log of diameter 20 cm. The figures show different ways this can be done.
(a) Express the cross-sectional area of the beam as a function of the angle θ in the figures.
(b) Graph the function you found in part (a).
(c) Find the dimensions of the beam with largest cross-sectional area.



66. Strength of a Beam The strength of a beam is proportional to the width and the square of the depth. A beam is cut from a log as in Exercise 65. Express the strength of the beam as a function of the angle θ in the figures.

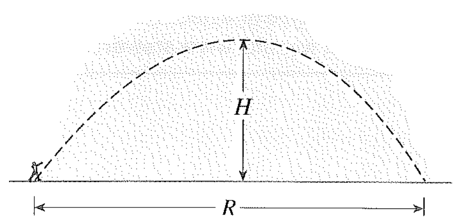
67. Throwing a Shot Put The range R and height H of a shot put thrown with an initial velocity of v_0 ft/s at an angle θ are given by

$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

$$H = \frac{v_0^2 \sin^2\theta}{2g}$$

On the earth $g = 32$ ft/s^2 and on the moon $g = 5.2$ ft/s^2. Find the range and height of a shot put thrown under the given conditions.

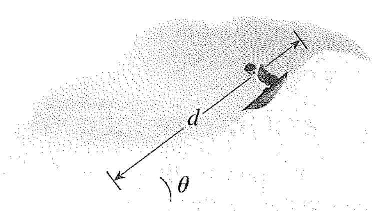
(a) On the earth with $v_0 = 12$ ft/s and $\theta = \pi/6$
(b) On the moon with $v_0 = 12$ ft/s and $\theta = \pi/6$



68. Sledding The time in seconds that it takes for a sled to slide down a hillside inclined at an angle θ is

$$t = \sqrt{\frac{d}{16 \sin \theta}}$$

where d is the length of the slope in feet. Find the time it takes to slide down a 2000-ft slope inclined at 30°.

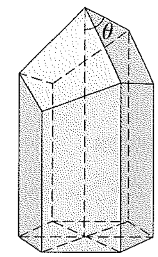


69. Beehives In a beehive each cell is a regular hexagonal prism, as shown in the figure. The amount of wax W in the cell depends on the apex angle θ and is given by

$$W = 3.02 - 0.38 \cot \theta + 0.65 \csc \theta$$

Bees instinctively choose θ so as to use the least amount of wax possible.

(a) Use a graphing device to graph W as a function of θ for $0 < \theta < \pi$.
(b) For what value of θ does W have its minimum value? [*Note:* Biologists have discovered that bees rarely deviate from this value by more than a degree or two.]

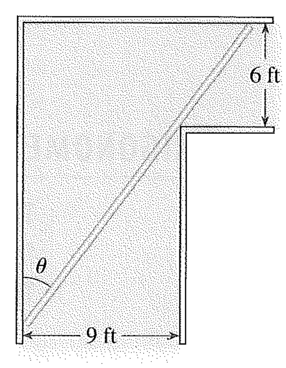


70. Turning a Corner A steel pipe is being carried down a hallway that is 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide.

(a) Show that the length of the pipe in the figure is modeled by the function

$$L(\theta) = 9 \csc \theta + 6 \sec \theta$$

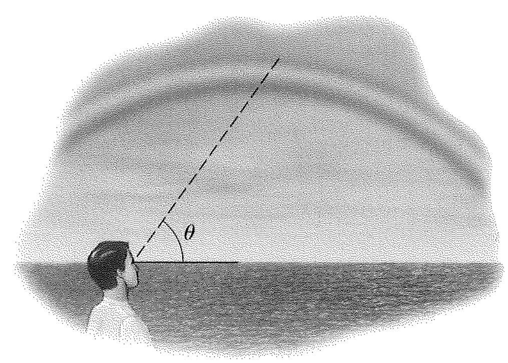
(b) Graph the function L for $0 < \theta < \pi/2$.
(c) Find the minimum value of the function L.
(d) Explain why the value of L you found in part (c) is the length of the longest pipe that can be carried around the corner.



71. Rainbows Rainbows are created when sunlight of different wavelengths (colors) is refracted and reflected in raindrops. The angle of elevation θ of a rainbow is always the same. It can be shown that $\theta = 4\beta - 2\alpha$, where

$$\sin \alpha = k \sin \beta$$

and $\alpha = 59.4°$ and $k = 1.33$ is the index of refraction of water. Use the given information to find the angle of elevation θ of a rainbow. (For a mathematical explanation of rainbows see *Calculus Early Transcendentals,* 7th Edition, by James Stewart, page 282.)



DISCOVERY ■ DISCUSSION ■ WRITING

72. **Using a Calculator** To solve a certain problem, you need to find the sine of 4 rad. Your study partner uses his calculator and tells you that

$$\sin 4 = 0.0697564737$$

On your calculator you get

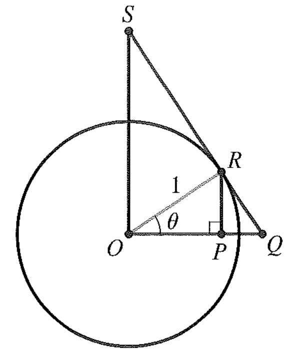
$$\sin 4 = -0.7568024953$$

What is wrong? What mistake did your partner make?

73. **Viète's Trigonometric Diagram** In the 16th century the French mathematician François Viète (see page 49) published the following remarkable diagram. Each of the six trigonometric functions of θ is equal to the length of a line segment in the figure. For instance, $\sin\theta = |PR|$, since from $\triangle OPR$ we see that

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{|PR|}{|OR|} = \frac{|PR|}{1} = |PR|$$

For each of the five other trigonometric functions, find a line segment in the figure whose length equals the value of the function at θ. (*Note:* The radius of the circle is 1, the center is O, segment QS is tangent to the circle at R, and $\angle SOQ$ is a right angle.)



DISCOVERY PROJECT **Similarity**

In this project we explore the idea of similarity and some of its consequences for any type of figure. You can find the project at the book companion website: **www.stewartmath.com**

6.4 Inverse Trigonometric Functions and Right Triangles

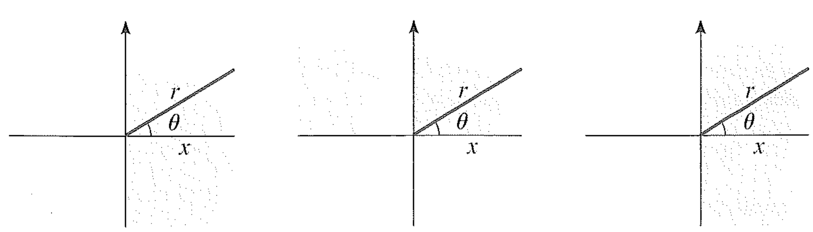
The Inverse Sine, Inverse Cosine, and Inverse Tangent Functions ▶ Solving for Angles in Right Triangles ▶ Evaluating Expressions Involving Inverse Trigonometric Functions

The graphs of the inverse trigonometric functions are studied in Section 5.5.

Recall that for a function to have an inverse, it must be one-to-one. Since the trigonometric functions are not one-to-one, they do not have inverses. So we restrict the domain of each of the trigonometric functions to intervals on which they attain all their values and on which they are one-to-one. The resulting functions have the same range as the original functions but are one-to-one.

▼ The Inverse Sine, Inverse Cosine, and Inverse Tangent Functions

Let's first consider the sine function. We restrict the domain of the sine function to angles θ with $-\pi/2 \le \theta \le \pi/2$. From Figure 1 we see that on this domain the sine function attains each of the values in the interval $[-1, 1]$ exactly once and so is one-to-one. Similarly, we restrict the domains of cosine and tangent as shown in Figure 1.



$\sin\theta = \frac{y}{r}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\qquad$ $\cos\theta = \frac{x}{r}$, $0 \le \theta \le \pi$ $\qquad$ $\tan\theta = \frac{x}{y}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

FIGURE 1 Restricted domains of the sine, cosine, and tangent functions