5.4 More Trigonometric Graphs

Graphs of Tangent, Cotangent, Secant, and Cosecant ▶ Graphs of Transformations of Tangent and Cotangent ▶ Graphs of Transformations of Cosecant and Secant

In this section we graph the tangent, cotangent, secant, and cosecant functions and transformations of these functions.

▼ Graphs of Tangent, Cotangent, Secant, and Cosecant

We begin by stating the periodic properties of these functions. Recall that sine and cosine have period 2π. Since cosecant and secant are the reciprocals of sine and cosine, respectively, they also have period 2π (see Exercise 55). Tangent and cotangent, however, have period π (see Exercise 85 of Section 5.2).

PERIODIC PROPERTIES

The functions tangent and cotangent have period π:

$$\tan(x + \pi) = \tan x \qquad \cot(x + \pi) = \cot x$$

The functions cosecant and secant have period 2π:

$$\csc(x + 2\pi) = \csc x \qquad \sec(x + 2\pi) = \sec x$$

x	$\tan x$
0	0
$\pi/6$	0.58
$\pi/4$	1.00
$\pi/3$	1.73
1.4	5.80
1.5	14.10
1.55	48.08
1.57	1,255.77
1.5707	10,381.33

We first sketch the graph of tangent. Since it has period π, we need only sketch the graph on any interval of length π and then repeat the pattern to the left and to the right. We sketch the graph on the interval $(-\pi/2, \pi/2)$. Since $\tan(\pi/2)$ and $\tan(-\pi/2)$ aren't defined, we need to be careful in sketching the graph at points near $\pi/2$ and $-\pi/2$. As x gets near $\pi/2$ through values less than $\pi/2$, the value of $\tan x$ becomes large. To see this, notice that as x gets close to $\pi/2$, $\cos x$ approaches 0 and $\sin x$ approaches 1 and so $\tan x = \sin x/\cos x$ is large. A table of values of $\tan x$ for x close to $\pi/2$ ($\approx$1.570796) is shown in the margin.

Thus, by choosing x close enough to $\pi/2$ through values less than $\pi/2$, we can make the value of $\tan x$ larger than any given positive number. We express this by writing

$$\tan x \to \infty \qquad \text{as} \qquad x \to \frac{\pi}{2}^{-}$$

This is read "$\tan x$ approaches infinity as x approaches $\pi/2$ from the left."

In a similar way, by choosing x close to $-\pi/2$ through values greater than $-\pi/2$, we can make $\tan x$ smaller than any given negative number. We write this as

Arrow notation is discussed in Section 3.7.

$$\tan x \to -\infty \qquad \text{as} \qquad x \to -\frac{\pi}{2}^{+}$$

This is read "$\tan x$ approaches negative infinity as x approaches $-\pi/2$ from the right."

Asymptotes are discussed in Section 3.7.

Thus, the graph of $y = \tan x$ approaches the vertical lines $x = \pi/2$ and $x = -\pi/2$. So these lines are **vertical asymptotes**. With the information we have so far, we sketch the graph of

$y = \tan x$ for $-\pi/2 < x < \pi/2$ in Figure 1. The complete graph of tangent (see Figure 5(a) on the next page) is now obtained using the fact that tangent is periodic with period π.

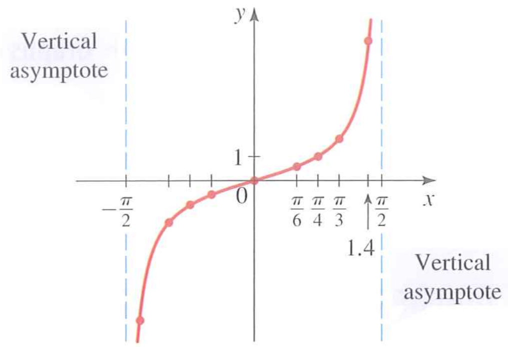


FIGURE 1
One period of $y = \tan x$

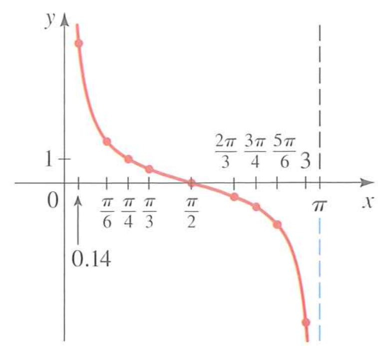


FIGURE 2
One period of $y = \cot x$

MATHEMATICS IN THE MODERN WORLD

Evaluating Functions on a Calculator

How does your calculator evaluate $\sin t$, $\cos t$, e^t, $\ln t$, $\sqrt{t}$, and other such functions? One method is to approximate these functions by polynomials, because polynomials are easy to evaluate. For example,

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

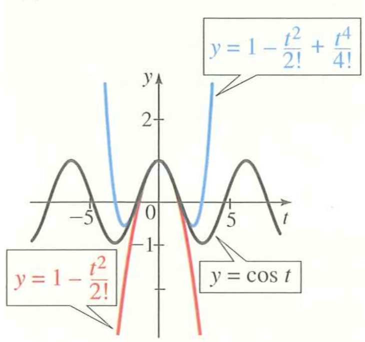
$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$

where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. These remarkable formulas were found by the British mathematician Brook Taylor (1685–1731). For instance, if we use the first three terms of Taylor's series to find cos(0.4), we get

$$\cos 0.4 \approx 1 - \frac{(0.4)^2}{2!} + \frac{(0.4)^4}{4!}$$

$$\approx 0.92106667$$

(Compare this with the value you get from your calculator.) The graph shows that the more terms of the series we use, the more closely the polynomials approximate the function $\cos t$.



The function $y = \cot x$ is graphed on the interval $(0, \pi)$ by a similar analysis (see Figure 2). Since $\cot x$ is undefined for $x = n\pi$ with n an integer, its complete graph (in Figure 5(b) on the next page) has vertical asymptotes at these values.

To graph the cosecant and secant functions, we use the reciprocal identities

$$\csc x = \frac{1}{\sin x} \qquad \text{and} \qquad \sec x = \frac{1}{\cos x}$$

So to graph $y = \csc x$, we take the reciprocals of the y-coordinates of the points of the graph of $y = \sin x$. (See Figure 3.) Similarly, to graph $y = \sec x$, we take the reciprocals of the y-coordinates of the points of the graph of $y = \cos x$. (See Figure 4.)

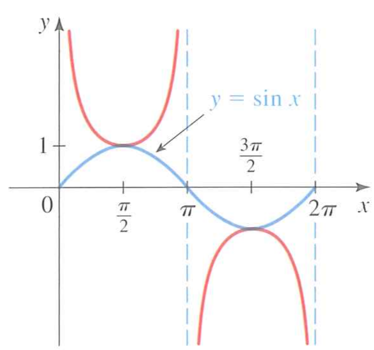


FIGURE 3
One period of $y = \csc x$

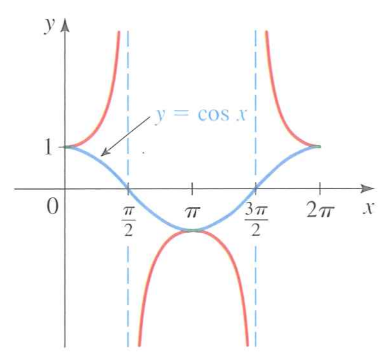


FIGURE 4
One period of $y = \sec x$

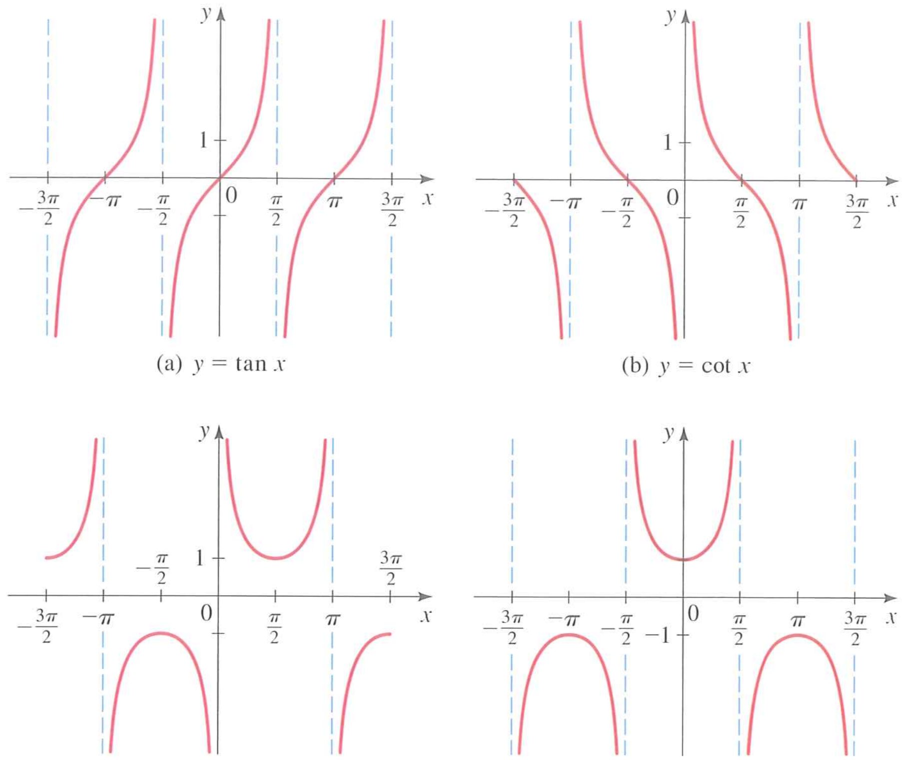
Let's consider more closely the graph of the function $y = \csc x$ on the interval $0 < x < \pi$. We need to examine the values of the function near 0 and π, since at these values $\sin x = 0$, and $\csc x$ is thus undefined. We see that

$$\csc x \to \infty \quad \text{as} \quad x \to 0^+$$

$$\csc x \to \infty \quad \text{as} \quad x \to \pi^-$$

Thus, the lines $x = 0$ and $x = \pi$ are vertical asymptotes. In the interval $\pi < x < 2\pi$ the graph is sketched in the same way. The values of $\csc x$ in that interval are the same as those in the interval $0 < x < \pi$ except for sign (see Figure 3). The complete graph in Figure 5(c) is now obtained from the fact that the function cosecant is periodic with period

2π. Note that the graph has vertical asymptotes at the points where $\sin x = 0$, that is, at $x = n\pi$, for n an integer.



(a) $y = \tan x$ (b) $y = \cot x$

(c) $y = \csc x$ (d) $y = \sec x$

FIGURE 5

The graph of $y = \sec x$ is sketched in a similar manner. Observe that the domain of $\sec x$ is the set of all real numbers other than $x = (\pi/2) + n\pi$, for n an integer, so the graph has vertical asymptotes at those points. The complete graph is shown in Figure 5(d).

It is apparent that the graphs of $y = \tan x$, $y = \cot x$, and $y = \csc x$ are symmetric about the origin, whereas that of $y = \sec x$ is symmetric about the y-axis. This is because tangent, cotangent, and cosecant are odd functions, whereas secant is an even function.

▼ Graphs of Transformations of Tangent and Cotangent

We now consider graphs of transformations of the tangent and cotangent functions.

EXAMPLE 1 | Graphing Tangent Curves

Graph each function.

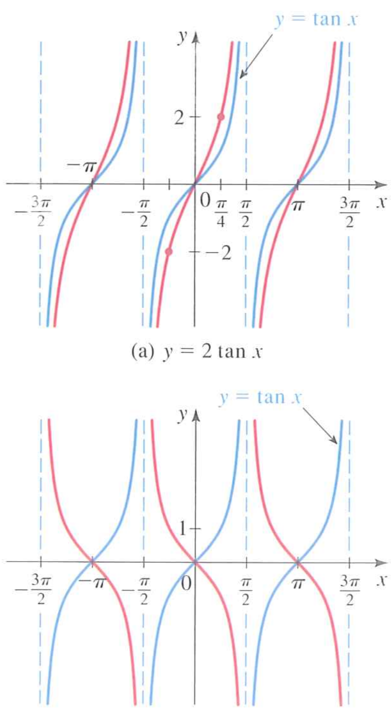
(a) $y = 2 \tan x$ **(b)** $y = -\tan x$

SOLUTION We first graph $y = \tan x$ and then transform it as required.

(a) To graph $y = 2 \tan x$, we multiply the y-coordinate of each point on the graph of $y = \tan x$ by 2. The resulting graph is shown in Figure 6(a).

(b) The graph of $y = -\tan x$ in Figure 6(b) is obtained from that of $y = \tan x$ by reflecting in the x-axis.

NOW TRY EXERCISES **9** AND **11**



(a) $y = 2 \tan x$

(b) $y = -\tan x$

FIGURE 6

Since the tangent and cotangent functions have period π, the functions

$$y = a \tan kx \qquad \text{and} \qquad y = a \cot kx \qquad (k > 0)$$

complete one period as kx varies from 0 to π, that is, for $0 \le kx \le \pi$. Solving this inequality, we get $0 \le x \le \pi/k$. So they each have period π/k.

TANGENT AND COTANGENT CURVES

The functions

$$y = a \tan kx \qquad \text{and} \qquad y = a \cot kx \qquad (k > 0)$$

have period π/k.

Thus, one complete period of the graphs of these functions occurs on any interval of length π/k. To sketch a complete period of these graphs, it's convenient to select an interval between vertical asymptotes:

To graph one period of $y = a \tan kx$, an appropriate interval is $\left(-\dfrac{\pi}{2k}, \dfrac{\pi}{2k}\right)$.

To graph one period of $y = a \cot kx$, an appropriate interval is $\left(0, \dfrac{\pi}{k}\right)$.

EXAMPLE 2 | Graphing Tangent Curves

Graph each function.

(a) $y = \tan 2x$ **(b)** $y = \tan 2\left(x - \dfrac{\pi}{4}\right)$

SOLUTION

(a) The period is $\pi/2$ and an appropriate interval is $(-\pi/4, \pi/4)$. The endpoints $x = -\pi/4$ and $x = \pi/4$ are vertical asymptotes. Thus, we graph one complete period of the function on $(-\pi/4, \pi/4)$. The graph has the same shape as that of the tangent function, but is shrunk horizontally by a factor of $\frac{1}{2}$. We then repeat that portion of the graph to the left and to the right. See Figure 7(a).

(b) The graph is the same as that in part (a), but it is shifted to the right $\pi/4$, as shown in Figure 7(b).

Since $y = \tan x$ completes one period between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, the function $y = \tan 2(x - \frac{\pi}{4})$ completes one period as $2(x - \frac{\pi}{4})$ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Start of period:	End of period:
$2(x - \frac{\pi}{4}) = -\frac{\pi}{2}$	$2(x - \frac{\pi}{4}) = \frac{\pi}{2}$
$x - \frac{\pi}{4} = -\frac{\pi}{4}$	$x - \frac{\pi}{4} = \frac{\pi}{4}$
$x = 0$	$x = \frac{\pi}{2}$

So we graph one period on the interval $(0, \frac{\pi}{2})$.

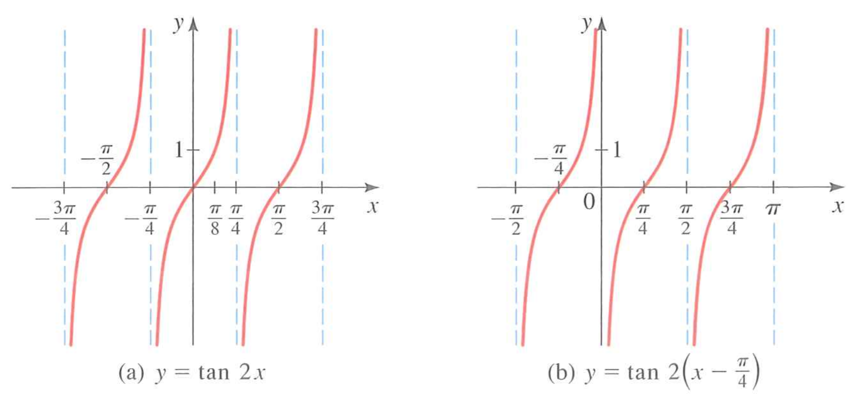


FIGURE 7

NOW TRY EXERCISES **27** AND **39**

EXAMPLE 3 | A Shifted Cotangent Curve

Graph $y = 2\cot\left(3x - \frac{\pi}{2}\right)$.

SOLUTION We first put this in the form $y = a\cot k(x - b)$ by factoring 3 from the expression $3x - \frac{\pi}{2}$:

$$y = 2\cot\left(3x - \frac{\pi}{2}\right) = 2\cot 3\left(x - \frac{\pi}{6}\right)$$

Since $y = \cot x$ completes one period between $x = 0$ and $x = \pi$, the function $y = 2\cot(3x - \frac{\pi}{2})$ completes one period as $3x - \frac{\pi}{2}$ varies from 0 to π.

Start of period:	End of period:
$3x - \frac{\pi}{2} = 0$	$3x - \frac{\pi}{2} = \pi$
$3x = \frac{\pi}{2}$	$3x = \frac{3\pi}{2}$
$x = \frac{\pi}{6}$	$x = \frac{\pi}{2}$

So we graph one period on the interval $(\frac{\pi}{6}, \frac{\pi}{2})$.

Thus the graph is the same as that of $y = 2\cot 3x$ but is shifted to the right $\pi/6$. The period of $y = 2\cot 3x$ is $\pi/3$, and an appropriate interval is $(0, \pi/3)$. To get the corresponding interval for the desired graph, we shift this interval to the right $\pi/6$. This gives

$$\left(0 + \frac{\pi}{6}, \frac{\pi}{3} + \frac{\pi}{6}\right) = \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$$

Finally, we graph one period in the shape of cotangent on the interval $(\pi/6, \pi/2)$ and repeat that portion of the graph to the left and to the right. (See Figure 8.)

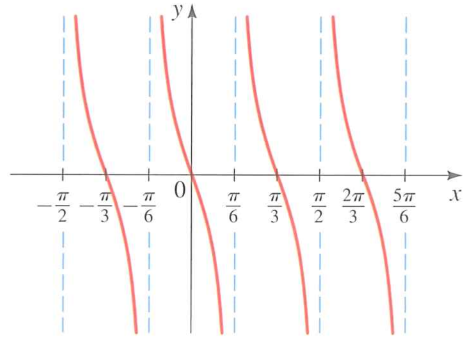


FIGURE 8
$y = 2\cot\left(3x - \frac{\pi}{2}\right)$

NOW TRY EXERCISE **43**

Graphs of Transformations of Cosecant and Secant

We have already observed that the cosecant and secant functions are the reciprocals of the sine and cosine functions. Thus, the following result is the counterpart of the result for sine and cosine curves in Section 5.3.

COSECANT AND SECANT CURVES

The functions

$$y = a\csc kx \quad \text{and} \quad y = a\sec kx \quad (k > 0)$$

have period $2\pi/k$.

An appropriate interval on which to graph one complete period is $[0, 2\pi/k]$.

EXAMPLE 4 | Graphing Cosecant Curves

Graph each function.

(a) $y = \frac{1}{2}\csc 2x$ **(b)** $y = \frac{1}{2}\csc\left(2x + \frac{\pi}{2}\right)$

SOLUTION

(a) The period is $2\pi/2 = \pi$. An appropriate interval is $[0, \pi]$, and the asymptotes occur in this interval whenever $\sin 2x = 0$. So the asymptotes in this interval are $x = 0$, $x = \pi/2$, and $x = \pi$. With this information we sketch on the interval $[0, \pi]$ a graph with the same general shape as that of one period of the cosecant function. The complete graph in Figure 9(a) is obtained by repeating this portion of the graph to the left and to the right.

Since $y = \csc x$ completes one period between $x = 0$ and $x = 2\pi$, the function $y = \frac{1}{2}\csc(2x + \frac{\pi}{2})$ completes one period as $2x + \frac{\pi}{2}$ varies from 0 to 2π.

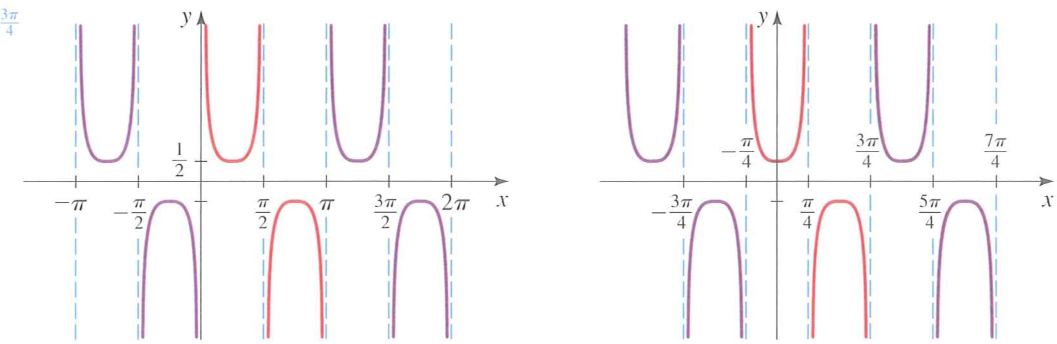
Start of period:	End of period:
$2x + \frac{\pi}{2} = 0$	$2x + \frac{\pi}{2} = 2\pi$
$2x = -\frac{\pi}{2}$	$2x = \frac{3\pi}{2}$
$x = -\frac{\pi}{4}$	$x = \frac{3\pi}{4}$

So we graph one period on the interval $(-\frac{\pi}{4}, \frac{3\pi}{4})$.

(b) We first write

$$y = \frac{1}{2}\csc\left(2x + \frac{\pi}{2}\right) = \frac{1}{2}\csc 2\left(x + \frac{\pi}{4}\right)$$

From this we see that the graph is the same as that in part (a) but shifted to the left $\pi/4$. The graph is shown in Figure 9(b).



(a) $y = \frac{1}{2}\csc 2x$

(b) $y = \frac{1}{2}\csc\left(2x + \frac{\pi}{2}\right)$

FIGURE 9

NOW TRY EXERCISES **33** AND **45**

EXAMPLE 5 | Graphing a Secant Curve

Graph $y = 3\sec\frac{1}{2}x$.

SOLUTION The period is $2\pi \div \frac{1}{2} = 4\pi$. An appropriate interval is $[0, 4\pi]$, and the asymptotes occur in this interval wherever $\cos\frac{1}{2}x = 0$. Thus, the asymptotes in this interval are $x = \pi$, $x = 3\pi$. With this information we sketch on the interval $[0, 4\pi]$ a graph with the same general shape as that of one period of the secant function. The complete graph in Figure 10 is obtained by repeating this portion of the graph to the left and to the right.

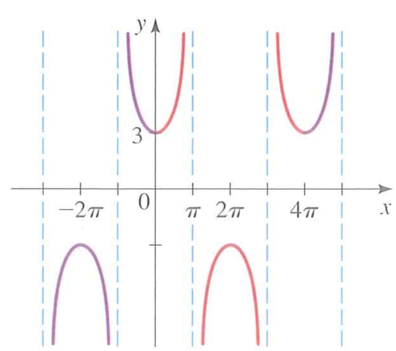


FIGURE 10
$y = 3\sec\frac{1}{2}x$

NOW TRY EXERCISE **31**

5.4 EXERCISES

CONCEPTS

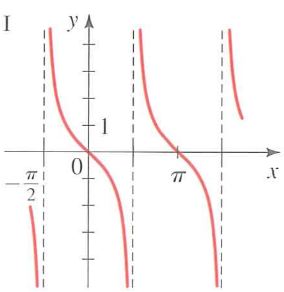
1. The trigonometric function $y = \tan x$ has period ________ and asymptotes $x =$ ________. Sketch a graph of this function on the interval $(-\pi/2, \pi/2)$.

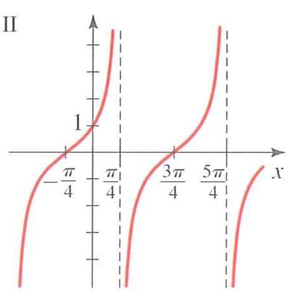
2. The trigonometric function $y = \csc x$ has period ________ and asymptotes $x =$ ________. Sketch a graph of this function on the interval $(-\pi, \pi)$.

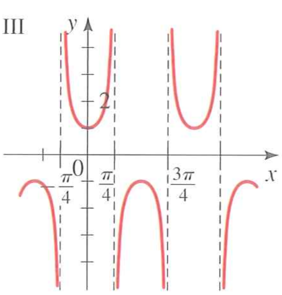
SKILLS

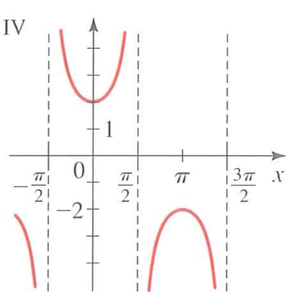
3–8 ■ Match the trigonometric function with one of the graphs I–VI.

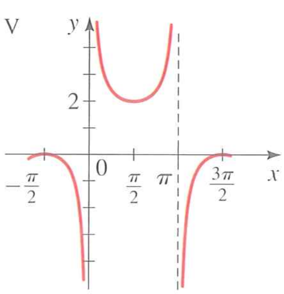
3. $f(x) = \tan\left(x + \dfrac{\pi}{4}\right)$
4. $f(x) = \sec 2x$
5. $f(x) = \cot 2x$
6. $f(x) = -\tan x$
7. $f(x) = 2 \sec x$
8. $f(x) = 1 + \csc x$

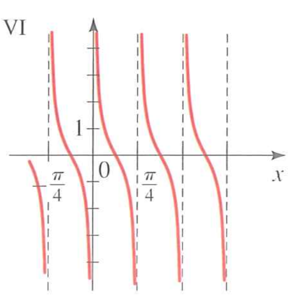
I


II


III


IV


V


VI


9–54 ■ Find the period and graph the function.

9. $y = 4 \tan x$
10. $y = -4 \tan x$
11. $y = -\frac{1}{2} \tan x$
12. $y = \frac{1}{2} \tan x$
13. $y = -\cot x$
14. $y = 2 \cot x$
15. $y = 2 \csc x$
16. $y = \frac{1}{2} \csc x$
17. $y = 3 \sec x$
18. $y = -3 \sec x$
19. $y = \tan\left(x + \dfrac{\pi}{2}\right)$
20. $y = \tan\left(x - \dfrac{\pi}{4}\right)$
21. $y = \csc\left(x - \dfrac{\pi}{2}\right)$
22. $y = \sec\left(x + \dfrac{\pi}{4}\right)$
23. $y = \cot\left(x + \dfrac{\pi}{4}\right)$
24. $y = 2 \csc\left(x - \dfrac{\pi}{3}\right)$
25. $y = \dfrac{1}{2} \sec\left(x - \dfrac{\pi}{6}\right)$
26. $y = 3 \csc\left(x + \dfrac{\pi}{2}\right)$
27. $y = \tan 4x$
28. $y = \tan \frac{1}{2}x$
29. $y = \tan \dfrac{\pi}{4}x$
30. $y = \cot \dfrac{\pi}{2}x$
31. $y = \sec 2x$
32. $y = 5 \csc 3x$
33. $y = \csc 4x$
34. $y = \csc \frac{1}{2}x$
35. $y = 2 \tan 3\pi x$
36. $y = 2 \tan \dfrac{\pi}{2}x$
37. $y = 5 \csc \dfrac{3\pi}{2}x$
38. $y = 5 \sec 2\pi x$
39. $y = \tan 2\left(x + \dfrac{\pi}{2}\right)$
40. $y = \csc 2\left(x + \dfrac{\pi}{2}\right)$
41. $y = \tan 2(x - \pi)$
42. $y = \sec 2\left(x - \dfrac{\pi}{2}\right)$
43. $y = \cot\left(2x - \dfrac{\pi}{2}\right)$
44. $y = \frac{1}{2} \tan(\pi x - \pi)$
45. $y = 2 \csc\left(\pi x - \dfrac{\pi}{3}\right)$
46. $y = 2 \sec\left(\dfrac{1}{2}x - \dfrac{\pi}{3}\right)$
47. $y = 5 \sec\left(3x - \dfrac{\pi}{2}\right)$
48. $y = \frac{1}{2} \sec(2\pi x - \pi)$
49. $y = \tan\left(\dfrac{2}{3}x - \dfrac{\pi}{6}\right)$
50. $y = \tan \dfrac{1}{2}\left(x + \dfrac{\pi}{4}\right)$
51. $y = 3 \sec \pi\left(x + \dfrac{1}{2}\right)$
52. $y = \sec\left(3x + \dfrac{\pi}{2}\right)$
53. $y = -2 \tan\left(2x - \dfrac{\pi}{3}\right)$
54. $y = 2 \csc(3x + 3)$

55. **(a)** Prove that if f is periodic with period p, then $1/f$ is also periodic with period p.
    (b) Prove that cosecant and secant each have period 2π.

56. Prove that if f and g are periodic with period p, then f/g is also periodic, but its period could be smaller than p.

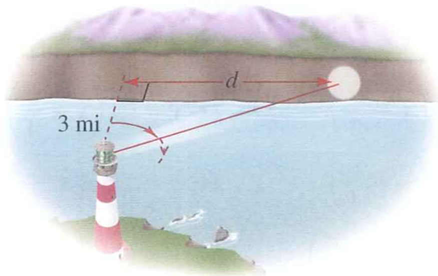
APPLICATIONS

57. **Lighthouse** The beam from a lighthouse completes one rotation every two minutes. At time t, the distance d shown in the figure on the next page is

$$d(t) = 3 \tan \pi t$$

where t is measured in minutes and d in miles.
(a) Find $d(0.15)$, $d(0.25)$, and $d(0.45)$.

(b) Sketch a graph of the function d for $0 \le t < \frac{1}{2}$.
(c) What happens to the distance d as t approaches $\frac{1}{2}$?

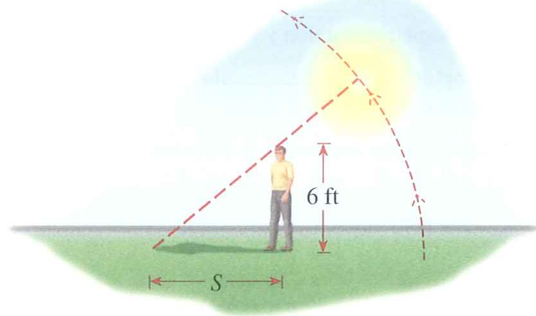


58. Length of a Shadow On a day when the sun passes directly overhead at noon, a six-foot-tall man casts a shadow of length

$$S(t) = 6\left|\cot \frac{\pi}{12}t\right|$$

where S is measured in feet and t is the number of hours since 6 A.M.

(a) Find the length of the shadow at 8:00 A.M., noon, 2:00 P.M., and 5:45 P.M.
(b) Sketch a graph of the function S for $0 < t < 12$.
(c) From the graph determine the values of t at which the length of the shadow equals the man's height. To what time of day does each of these values correspond?
(d) Explain what happens to the shadow as the time approaches 6 P.M. (that is, as $t \to 12^-$).



DISCOVERY ■ DISCUSSION ■ WRITING

59. Reduction Formulas Use the graphs in Figure 5 to explain why the following formulas are true.

$$\tan\left(x - \frac{\pi}{2}\right) = -\cot x$$

$$\sec\left(x - \frac{\pi}{2}\right) = \csc x$$

5.5 Inverse Trigonometric Functions and Their Graphs

The Inverse Sine Function ▶ The Inverse Cosine Function ▶ The Inverse Tangent Function ▶ The Inverse Secant, Cosecant, and Cotangent Functions

We study applications of inverse trigonometric functions to triangles in Sections 6.4–6.6.

Recall from Section 2.7 that the inverse of a function f is a function f^{-1} that reverses the rule of f. For a function to have an inverse, it must be one-to-one. Since the trigonometric functions are not one-to-one, they do not have inverses. It is possible, however, to restrict the domains of the trigonometric functions in such a way that the resulting functions are one-to-one.

▼ The Inverse Sine Function

Let's first consider the sine function. There are many ways to restrict the domain of sine so that the new function is one-to-one. A natural way to do this is to restrict the domain to the interval $[-\pi/2, \pi/2]$. The reason for this choice is that sine is one-to-one on this interval and moreover attains each of the values in its range on this interval. From Figure 1 we see that sine is one-to-one on this restricted domain (by the Horizontal Line Test) and so has an inverse.

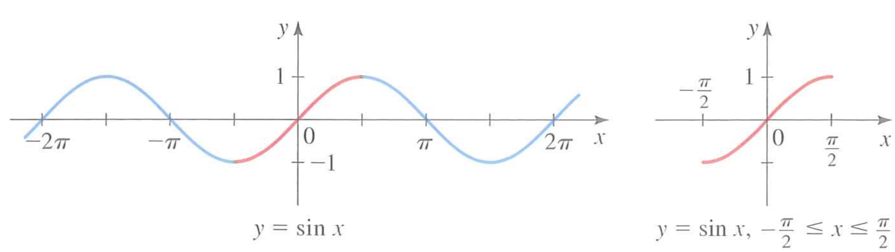


FIGURE 1 Graphs of the sine function and the restricted sine function