

TRIGONOMETRIC FUNCTIONS: UNIT CIRCLE APPROACH

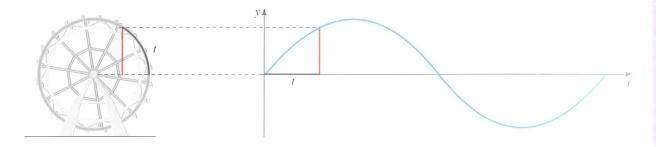
- **5.1** The Unit Circle
- **5.2** Trigonometric Functions of Real Numbers
- **5.3** Trigonometric Graphs
- **5.4** More Trigonometric Graphs
- **5.5** Inverse Trigonometric Functions and Their Graphs
- **5.6** Modeling Harmonic Motion

FOCUS ON MODELING

Fitting Sinusoidal Curves to Data

If you've ever taken a ferris wheel ride, then you know about periodic motion—that is, motion that repeats over and over. Periodic motion is common in nature. Think about the daily rising and setting of the sun (day, night, day, night, . . .), the daily variation in tide levels (high, low, high, low, . . .), or the vibrations of a leaf in the wind (left, right, left, right, . . .). To model such motion, we need a function whose values increase, then decrease, then increase, and so on. To understand how to define such a function, let's look at a person riding on a ferris wheel. The graph shows the height of the person above the center of the wheel at time *t*. Notice that the graph goes up and down repeatedly.

The trigonometric function *sine* is defined in a similar way, using the unit circle (in place of the ferris wheel). The trigonometric functions can be defined in two different but equivalent ways: as functions of real numbers (Chapter 5) or as functions of angles (Chapter 6). The two approaches are independent of each other, so either Chapter 5 or Chapter 6 may be studied first. We study both approaches because the different approaches are required for different applications.



5.1 THE UNIT CIRCLE

The Unit Circle ► Terminal Points on the Unit Curcle ► The Reference Number

In this section we explore some properties of the circle of radius 1 centered at the origin. These properties are used in the next section to define the trigonometric functions.

▼ The Unit Circle

The set of points at a distance 1 from the origin is a circle of radius 1 (see Figure 1). In Section 1.8 we learned that the equation of this circle is $x^2 + y^2 = 1$.

THE UNIT CIRCLE

The **unit circle** is the circle of radius 1 centered at the origin in the *xy*-plane. Its equation is

$$x^2 + y^2 = 1$$

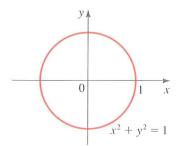


FIGURE 1 The unit circle

EXAMPLE 1 A Point on the Unit Circle

Show that the point $P\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right)$ is on the unit circle.

SOLUTION We need to show that this point satisfies the equation of the unit circle, that is, $x^2 + y^2 = 1$. Since

$$\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 = \frac{3}{9} + \frac{6}{9} = 1$$

P is on the unit circle.

NOW TRY EXERCISE 3

EXAMPLE 2 | Locating a Point on the Unit Circle

The point $P(\sqrt{3}/2, y)$ is on the unit circle in Quadrant IV. Find its y-coordinate.

SOLUTION Since the point is on the unit circle, we have

$$\left(\frac{\sqrt{3}}{2}\right)^{2} + y^{2} = 1$$

$$y^{2} = 1 - \frac{3}{4} = \frac{1}{4}$$

$$y = \pm \frac{1}{2}$$

Since the point is in Quadrant IV, its y-coordinate must be negative, so $y = -\frac{1}{2}$.

NOW TRY EXERCISE 9

▼ Terminal Points on the Unit Circle

Suppose t is a real number. Let's mark off a distance t along the unit circle, starting at the point (1,0) and moving in a counterclockwise direction if t is positive or in a clockwise direction if t is negative (Figure 2). In this way we arrive at a point P(x, y) on the unit circle.

cle. The point P(x, y) obtained in this way is called the **terminal point** determined by the real number t.

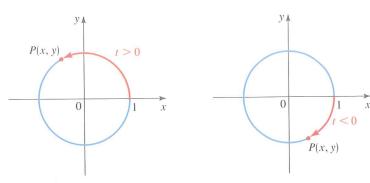


FIGURE 2

(a) Terminal point P(x, y) determined by t > 0

(b) Terminal point P(x, y) determined

The circumference of the unit circle is $C = 2\pi(1) = 2\pi$. So if a point starts at (1,0)and moves counterclockwise all the way around the unit circle and returns to (1,0), it travels a distance of 2π . To move halfway around the circle, it travels a distance of $\frac{1}{2}(2\pi) = \pi$. To move a quarter of the distance around the circle, it travels a distance of $\frac{1}{4}(2\pi) = \pi/2$. Where does the point end up when it travels these distances along the circle? From Figure 3 we see, for example, that when it travels a distance of π starting at (1,0), its terminal point is (-1,0).

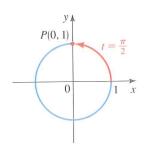
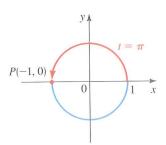
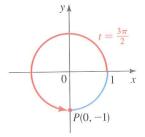
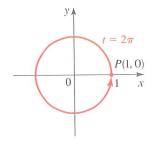


FIGURE 3 Terminal points determined by $t = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ and } 2\pi$







EXAMPLE 3 | Finding Terminal Points

Find the terminal point on the unit circle determined by each real number t.

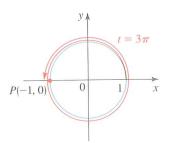
(a)
$$t = 3\pi$$

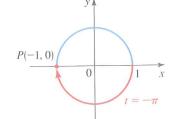
(b)
$$t = -\pi$$

(c)
$$t = -\frac{\pi}{2}$$

SOLUTION From Figure 4 we get the following:

- (a) The terminal point determined by 3π is (-1, 0).
- (b) The terminal point determined by $-\pi$ is (-1, 0).
- (c) The terminal point determined by $-\pi/2$ is (0, -1).





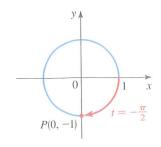


FIGURE 4

Notice that different values of t can determine the same terminal point.

from (0, 1) along the unit circle (see Figure 5).

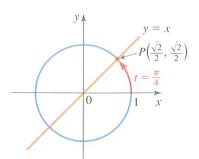


FIGURE 5

Since the unit circle is symmetric with respect to the line y = x, it follows that P lies on the line y = x. So P is the point of intersection (in the first quadrant) of the circle $x^2 + y^2 = 1$ and the line y = x. Substituting x for y in the equation of the circle, we get

$$x^{2} + x^{2} = 1$$
 $2x^{2} = 1$
Combine like terms
 $x^{2} = \frac{1}{2}$
Divide by 2
$$x = \pm \frac{1}{\sqrt{2}}$$
Take square roots

Since P is in the first quadrant, $x = 1/\sqrt{2}$ and since y = x, we have $y = 1/\sqrt{2}$ also. Thus, the terminal point determined by $\pi/4$ is

$$P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Similar methods can be used to find the terminal points determined by $t = \pi/6$ and $t = \pi/3$ (see Exercises 57 and 58). Table 1 and Figure 6 give the terminal points for some special values of t.

TABLE 1

Terminal point determined by t
(1,0)
$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
(0,1)

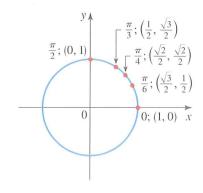


FIGURE 6

EXAMPLE 4 Finding Terminal Points

Find the terminal point determined by each given real number t.

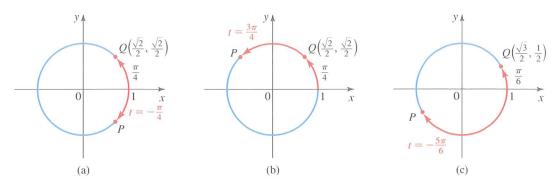
(a)
$$t = -\frac{\pi}{4}$$
 (b) $t = \frac{3\pi}{4}$ (c) $t = -\frac{5\pi}{6}$

(b)
$$t = \frac{3\pi}{4}$$

(c)
$$t = -\frac{5\pi}{6}$$

SOLUTION

(a) Let P be the terminal point determined by $-\pi/4$, and let Q be the terminal point determined by $\pi/4$. From Figure 7(a) we see that the point P has the same coordinates as Q except for sign. Since P is in quadrant IV, its x-coordinate is positive and its y-coordinate is negative. Thus, the terminal point is $P(\sqrt{2}/2, -\sqrt{2}/2)$.



- FIGURE 7
- (b) Let P be the terminal point determined by $3\pi/4$, and let Q be the terminal point determined by $\pi/4$. From Figure 7(b) we see that the point P has the same coordinates as Q except for sign. Since P is in quadrant II, its x-coordinate is negative and its y-coordinate is positive. Thus, the terminal point is $P(-\sqrt{2}/2, \sqrt{2}/2)$.
- (c) Let P be the terminal point determined by $-5\pi/6$, and let Q be the terminal point determined by $\pi/6$. From Figure 7(c) we see that the point P has the same coordinates as Q except for sign. Since P is in quadrant III, its coordinates are both negative. Thus, the terminal point is $P(-\sqrt{3}/2, -\frac{1}{2})$.
- NOW TRY EXERCISE 25

▼ The Reference Number

From Examples 3 and 4 we see that to find a terminal point in any quadrant we need only know the "corresponding" terminal point in the first quadrant. We use the idea of the reference number to help us find terminal points.

REFERENCE NUMBER

Let t be a real number. The **reference number** \bar{t} associated with t is the shortest distance along the unit circle between the terminal point determined by t and the x-axis.

Figure 8 shows that to find the reference number \bar{t} , it's helpful to know the quadrant in which the terminal point determined by t lies. If the terminal point lies in quadrants I or IV, where x is positive, we find \bar{t} by moving along the circle to the positive x-axis. If it lies in quadrants II or III, where x is negative, we find \bar{t} by moving along the circle to the negative x-axis.

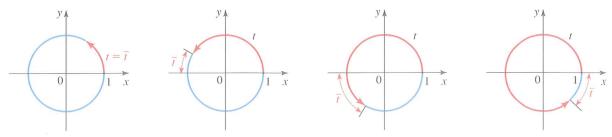


FIGURE 8 The reference number \bar{t} for t

EXAMPLE 5 | Finding Reference Numbers

Find the reference number for each value of t.

$$(a) t = \frac{5\pi}{6}$$

(b)
$$t = \frac{7\pi}{4}$$

(a)
$$t = \frac{5\pi}{6}$$
 (b) $t = \frac{7\pi}{4}$ (c) $t = -\frac{2\pi}{3}$ (d) $t = 5.80$

d)
$$t = 5.80$$

SOLUTION From Figure 9 we find the reference numbers as follows:

(a)
$$\bar{t} = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$$

(b)
$$\bar{t} = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$$

(c)
$$\bar{t} = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$$

(d)
$$\bar{t} = 2\pi - 5.80 \approx 0.48$$

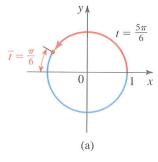
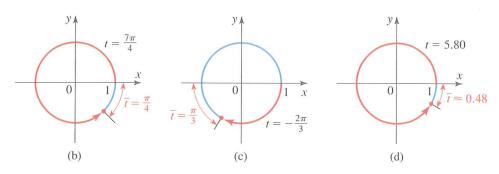


FIGURE 9



NOW TRY EXERCISE 35

USING REFERENCE NUMBERS TO FIND TERMINAL POINTS

To find the terminal point P determined by any value of t, we use the following

- **1.** Find the reference number \bar{t} .
- **2.** Find the terminal point Q(a, b) determined by \bar{t} .
- **3.** The terminal point determined by t is $P(\pm a, \pm b)$, where the signs are chosen according to the quadrant in which this terminal point lies.

EXAMPLE 6 | Using Reference Numbers to Find Terminal Points

Find the terminal point determined by each given real number t.

(a)
$$t = \frac{5\pi}{6}$$

(b)
$$t = \frac{7\pi}{4}$$

(b)
$$t = \frac{7\pi}{4}$$
 (c) $t = -\frac{2\pi}{3}$

SOLUTION The reference numbers associated with these values of t were found in Example 5.

(a) The reference number is $\bar{t} = \pi/6$, which determines the terminal point $(\sqrt{3}/2, \frac{1}{2})$ from Table 1. Since the terminal point determined by t is in Quadrant II, its x-coordinate is negative and its y-coordinate is positive. Thus, the desired terminal point is

$$\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$$

(b) The reference number is $\bar{t} = \pi/4$, which determines the terminal point $(\sqrt{2}/2, \sqrt{2}/2)$ from Table 1. Since the terminal point is in Quadrant IV, its x-coordinate is positive and its y-coordinate is negative. Thus, the desired terminal point is

$$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

(c) The reference number is $\bar{t} = \pi/3$, which determines the terminal point $(\frac{1}{2}, \sqrt{3}/2)$ from Table 1. Since the terminal point determined by t is in Quadrant III, its coordinates are both negative. Thus, the desired terminal point is

$$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

NOW TRY EXERCISE 39

Since the circumference of the unit circle is 2π , the terminal point determined by t is the same as that determined by $t+2\pi$ or $t-2\pi$. In general, we can add or subtract 2π any number of times without changing the terminal point determined by t. We use this observation in the next example to find terminal points for large t.

EXAMPLE 7 Finding the Terminal Point for Large t

Find the terminal point determined by $t = \frac{29\pi}{6}$.

SOLUTION Since

$$t = \frac{29\pi}{6} = 4\pi + \frac{5\pi}{6}$$

we see that the terminal point of t is the same as that of $5\pi/6$ (that is, we subtract 4π). So by Example 6(a) the terminal point is $(-\sqrt{3}/2, \frac{1}{2})$. (See Figure 10.)

NOW TRY EXERCISE 45

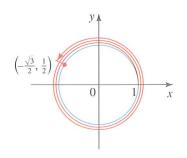


FIGURE 10

5.1 EXERCISES

CONCEPTS

- 1. (a) The unit circle is the circle centered at _____ with radius __
 - (b) The equation of the unit circle is ___
 - (c) Suppose the point P(x, y) is on the unit circle. Find the missing coordinate:

(i)
$$P(1,)$$

(ii)
$$P(-,1)$$

(iii)
$$P(-1,)$$

(iv)
$$P(-,-1)$$

- 2. (a) If we mark off a distance t along the unit circle, starting at (1, 0) and moving in a counterclockwise direction, we _____ point determined by t.
 - (b) The terminal points determined by $\pi/2$, π , $-\pi/2$, 2π ____, ______, and ___ respectively.

SKILLS

3–8 ■ Show that the point is on the unit circle.



	(5	12	1
4.	(-	13'	13	

5.
$$\left(\frac{7}{25}, \frac{24}{25}\right)$$

4.
$$\left(-\frac{5}{13}, \frac{12}{13}\right)$$
 5. $\left(\frac{7}{25}, \frac{24}{25}\right)$ **6.** $\left(-\frac{5}{7}, -\frac{2\sqrt{6}}{7}\right)$ **7.** $\left(-\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$ **8.** $\left(\frac{\sqrt{11}}{6}, \frac{5}{6}\right)$

7.
$$\left(-\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$$

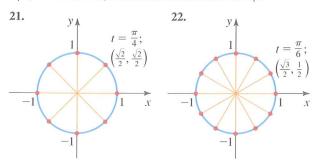
8.
$$\left(\frac{\sqrt{11}}{6}, \frac{5}{6}\right)$$

9–14 ■ Find the missing coordinate of P, using the fact that P lies on the unit circle in the given quadrant.

Coordinates	Quadrant
• 9. $P(-\frac{3}{5},$	III
10. $P(, -\frac{7}{25})$	IV
11. $P(, \frac{1}{3})$	II

Coordinates	Quadrant
12. $P(\frac{2}{5},)$	I
13. $P(, -\frac{2}{7})$	IV
14. $P(-\frac{2}{3},$	II

- **15–20** The point P is on the unit circle. Find P(x, y) from the given information.
- **15.** The *x*-coordinate of *P* is $\frac{4}{5}$, and the *y*-coordinate is positive.
- **16.** The y-coordinate of P is $-\frac{1}{3}$, and the x-coordinate is positive.
- 17. The y-coordinate of P is $\frac{2}{3}$, and the x-coordinate is negative.
- **18.** The *x*-coordinate of *P* is positive, and the *y*-coordinate of *P* is $-\sqrt{5}/5$.
- 19. The x-coordinate of P is $-\sqrt{2}/3$, and P lies below the x-axis.
- **20.** The x-coordinate of P is $-\frac{2}{5}$, and P lies above the x-axis.
- **21–22** Find t and the terminal point determined by t for each point in the figure. In Exercise 21, t increases in increments of $\pi/4$; in Exercise 22, t increases in increments of $\pi/6$.



23–32 ■ Find the terminal point P(x, y) on the unit circle determined by the given value of t.

$$23. t = \frac{\pi}{2}$$

24.
$$t = \frac{3\pi}{2}$$

$$25. t = \frac{5\pi}{6}$$

26.
$$t = \frac{7\pi}{6}$$

27.
$$t = -\frac{\pi}{3}$$

28.
$$t = \frac{5\pi}{3}$$

29.
$$t = \frac{2\pi}{3}$$

30.
$$t = -\frac{\pi}{2}$$

31.
$$t = -\frac{3\pi}{4}$$

32.
$$t = \frac{11\pi}{6}$$

33. Suppose that the terminal point determined by t is the point $\left(\frac{3}{5}, \frac{4}{5}\right)$ on the unit circle. Find the terminal point determined by each of the following.

(a)
$$\pi - t$$

(b)
$$-t$$

(c)
$$\pi + t$$

(d)
$$2\pi + t$$

34. Suppose that the terminal point determined by *t* is the point $(\frac{3}{4}, \sqrt{7}/4)$ on the unit circle. Find the terminal point determined by each of the following.

(a)
$$-t$$

(b)
$$4\pi + t$$

(c)
$$\pi - t$$

(d)
$$t - \pi$$

35–38 ■ Find the reference number for each value of t.

$$35.$$
 (a) $t = \frac{5\pi}{4}$

(b)
$$t = \frac{7\pi}{3}$$

(c)
$$t = -\frac{4\pi}{3}$$

(d)
$$t = \frac{\pi}{6}$$

36. (a)
$$t = \frac{5\pi}{6}$$

(b)
$$t = \frac{7\pi}{6}$$

(c)
$$t = \frac{11\pi}{3}$$

$$(\mathbf{d}) \ t = -\frac{7\pi}{4}$$

37. (a)
$$t = \frac{5\pi}{7}$$

(b)
$$t = -\frac{7\pi}{9}$$

(c)
$$t = -3$$

(d)
$$t = 5$$

38. (a)
$$t = \frac{11\pi}{5}$$

(b)
$$t = -\frac{9\pi}{7}$$

(c)
$$t = 6$$

(d)
$$t = -7$$

39–52 ■ Find (a) the reference number for each value of t and (b) the terminal point determined by t.

39.
$$t = \frac{2\pi}{3}$$

40.
$$t = \frac{4\pi}{3}$$

41.
$$t = \frac{3\pi}{4}$$

42.
$$t = \frac{7\pi}{3}$$

43.
$$t = -\frac{2\pi}{3}$$

44.
$$t = -\frac{7\pi}{6}$$

• 45.
$$t = \frac{13\pi}{4}$$

46.
$$t = \frac{13\pi}{6}$$

47.
$$t = \frac{7\pi}{6}$$

48.
$$t = \frac{17\pi}{4}$$

49.
$$t = -\frac{11\pi}{3}$$

50.
$$t = \frac{31\pi}{6}$$

51.
$$t = \frac{16\pi}{3}$$

52.
$$t = -\frac{41\pi}{4}$$

53–56 ■ Use the figure to find the terminal point determined by the real number t, with coordinates rounded to one decimal place.

53.
$$t = 1$$

54.
$$t = 2.5$$

55.
$$t = -1.1$$

56.
$$t = 4.2$$

